

Proper factorization theorems in high-energy scattering near the endpoint

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ABSTRACT: Consistent factorization theorems in high-energy scattering near the endpoint are presented in the framework of the soft-collinear effective theory. Conventional factorization theorem separates the soft and collinear parts successfully, but each part encounters infrared divergence and mixed ultraviolet and infrared divergences. We present factorization theorems in which the infrared divergences appear only in the parton distribution functions and the mixed divergence is removed by carefully separating and reorganizing collinear and soft parts. The underlying physical idea is to isolate and remove the soft contributions systematically from the collinear part in loop corrections order by order. After this procedure, each factorized term in the scattering cross sections is free of infrared divergence, and can be safely computed using perturbation theory. This factorization procedure can be applied to various high-energy scattering processes. We show factorization theorems in Drell-Yan processes, deep inelastic scattering and Higgs production near the endpoint.

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1 Introduction

Theoretical predictions on high-energy scattering are based on the factorization theorem, in which the cross section is factorized into the hard, collinear and soft parts. It means that, though the strong interaction affects the scattering processes at different scales in various stages, these effects can be separated according to their kinematic regimes. The hard part consists of the partonic cross sections with hard momenta. The collinear part describes the effects of the collimated energetic particles. For hadron colliders, the collinear part involves the parton distribution functions (PDF) in the initial state. If an exclusive process is considered, say, a pion production, there appears the fragmentation function which depicts the hadronization process from a parton into a hadron. This is also classified as the collinear part. The soft part describes soft gluon exchanges among different collinear sectors. These three parts are decoupled and there is no communication among them. Therefore it is possible to compute the contributions for each part separately using perturbative quantum chromodynamics (QCD).

The factorization property in various high-energy scattering processes has been well established in full QCD [1, 2]. In full QCD, once the scattering amplitudes are written,

they are examined closely by dividing collinear, soft momentum regions. A laborious analysis enables the factorization proof that collinear and soft parts are decoupled. The factorization proof becomes more elaborate in soft-collinear effective theory (SCET) [3–5] since the separation of collinear and soft parts is established from the outset at the operator level. The scattering cross sections are written as a product of the hard coefficient in terms of the Wilson coefficients, and the collinear and soft parts, which are expressed as the matrix elements of gauge-invariant operators. The factorization of the hard, collinear and soft parts in the hard scattering is now on a firm basis. We call this development the conventional factorization.

Since the advent of Large Hadron Collider (LHC), precise theoretical predictions on high-energy scattering can be under experimental scrutiny. As well as the fixed higher-order corrections in the strong coupling α_s , the resummed results using the renormalization group technique are available when there are large logarithms [1, 2, 6–9]. The theoretical accuracy available for the comparison with experiment can be attained by computing the factorized parts at higher orders or resumming large logarithms. The conventional factorization theorems have been successful because the collinear and soft parts can be expressed in terms of gauge-invariant operators and there is no communication between them. However, the conventional factorization theorems are plagued by infrared divergences.

In massless gauge theories like QCD, there exist infrared (IR) divergence as well as ultraviolet (UV) divergence. When a massless quark emits a gluon, collinear divergence appears when these two particles are collinear. Soft divergence emerges when the energy of the emitted gluon becomes soft. If the emitted gluon is both soft and collinear to the quark, soft and collinear divergences occur simultaneously. This type of divergence is referred to as the IR divergence. The UV divergence also shows up, but they can be removed by counter terms. The remnant after removing the UV divergence governs the scaling behavior. On the other hand, the IR divergences cannot be removed by hand. Instead, physical quantities should be free of IR divergence. The Kinoshita-Lee-Nauenberg (KLN) theorem [10, 11] succinctly states that the IR divergences appearing in real gluon emissions and in virtual corrections at any given order in α_s should cancel in scattering cross sections.

There is no IR divergence in the hard part, and most of the IR divergences are cancelled in the sum of the soft and collinear parts. The exception is the radiative correction to the PDF. But since the PDF is basically a nonperturbative quantity, the IR divergence can be absorbed in the PDF. However, as we will explain, the soft and collinear parts in the conventional factorization theorems contain IR divergences, and furthermore mixing between IR and UV divergences. Therefore the collinear and soft parts themselves cannot be regarded as physical quantities. The evolution of the collinear and soft parts via the anomalous dimension does not make sense either because these parts contain IR divergences and mixed divergences, hence the renormalization scale dependence does not stem from UV divergences.

Note that the IR divergences are not artifacts in calculating radiative corrections, and they always appear irrespective of the regularization methods. If we use the dimensional regularization to regulate both UV and IR divergences, the UV (IR) divergences appear as

poles of ϵ_{UV} (ϵ_{IR}). If the UV divergence is regulated by the dimensional regularization, and the IR divergence is regulated by the offshellness of external particles, the UV divergence still appears as poles in ϵ_{UV} , but the IR divergence appears as the logarithms of the offshellness. Unless the IR divergences are taken care of, the physical interpretation of each part with IR divergences is not possible, and we need more than the conventional factorization method.

Our new idea of factorization respects the philosophy of the conventional factorization theorems in the sense that the cross sections are factorized and the collinear and soft parts can be expressed in terms of the gauge-invariant operators. But it goes further by reorganizing the collinear and soft parts in such a way that the IR divergence resides only in the PDF, as in full QCD, and the remaining parts in the factorization formula are free of IR divergences. Only after this reorganization, the IR-finite quantities can be computed systematically in perturbation theory. However, this new factorization is not just reorganizing the factorized parts to shuffle IR divergences as a matter of convenience, but there is a deeper physical reason. In loop calculations, the collinear part unavoidably covers the kinematic region with soft momentum when the loop momentum becomes soft. But the soft region is already accounted for in the soft part. Therefore the soft contribution in the collinear part should be removed in an appropriate way to avoid double counting.

This removal can be systematically performed in SCET. The systematic technique to remove the soft contribution consistently from the collinear part is called the zero-bin subtraction [12] in SCET. The basic idea of the zero-bin subtraction is that, for each collinear Feynman diagram, the soft contribution can be computed by taking the limit where the loop momentum becomes soft, and then this soft contribution is subtracted from the naive collinear loop calculation. The zero-bin subtraction is not an esoteric technique used in SCET, but a simple tool to avoid double counting. The same idea can also be realized in full QCD. In full QCD, a careful separation of kinematic regions with different momenta should be performed, but the extraction of the soft contribution from the collinear part is tedious. SCET is an effective theory best fit to this purpose. SCET involves all the relevant fields necessary in the factorization, hence the factorization proof in SCET is equivalent to that in full QCD itself, but with more transparency.

We emphasize that care should be taken to perform an appropriate zero-bin subtraction. Naively any soft limit of the collinear loop momentum may do for calculating the zero-bin contributions, but in fact the soft limit of the collinear momentum should be determined by the size of the momentum entering the soft part. This is the essential point in introducing the zero-bin subtraction. If Q is the large scale in the system, and λ is a small parameter in SCET, the soft part describes the fluctuations of order $Q\lambda$ or $Q\lambda^2$, and the soft limit in the collinear part should also be of the same order. If the soft part describes the interaction with momentum of order $Q\lambda$ ($Q\lambda^2$), the soft limit in the collinear part should also be of order $Q\lambda$ ($Q\lambda^2$). If the soft momentum in the soft part is of order $Q\lambda$, while the soft limit in the collinear part is of order $Q\lambda^2$, or vice versa, there is a mismatch between the soft part and the soft contribution in the collinear part. With this mismatch, the zero-bin subtraction does not produce any sensible results, and the double counting is not removed completely.

After the double counting problem is handled appropriately, the scattering cross sections near the endpoint again factorize, but in a different form. They are factored into the hard part, the collinear part, and the soft kernel which consists of the original soft contributions and the removed soft part from the collinear part through the zero-bin subtraction. Then each term except the PDF is IR finite, and the resummation of large logarithms can be performed using the renormalization group equation. We show the factorization theorems in this picture in deep inelastic scattering (DIS), Drell-Yan (DY) processes near the endpoint, and present the one-loop results explicitly. This factorization proof also applies in computing resummation near the partonic threshold in the Higgs production, and it forms a robust basis in performing a systematic analysis of the threshold resummation.

The structure of the paper is as follows: In Sec. 2, the features of SCET are described briefly and the factorization theorems in DY, DIS processes near the endpoint are outlined. In Sec. 2.1, the detailed factorization theorem is presented for DY processes, and the factorization theorem for DIS near the endpoint is shown in Sec. 2.2. The factorization theorem for the Higgs production near the partonic threshold is presented in Sec. 2.3. In Sec. 3, the soft functions for all these processes are computed at one loop in terms of the matrix elements of the soft Wilson lines. The soft functions at higher orders clearly show that they include IR divergence and mixing between UV and IR divergences, hence not physical in themselves. In Sec. 4, we introduce the collinear distribution functions and the PDF as the same matrix elements of gauge-invariant collinear operators, but defined at different scales depending on the size of the soft momentum involved. Then these functions are computed with the corresponding zero-bin subtractions. Here we also explain why the need for the zero-bin subtraction in computing the PDF has remained unnoticed in full QCD and in SCET. In Sec. 5, the jet function, which describes the collinear particles in the final state in DIS, is computed to one loop including the zero-bin subtraction. In Sec. 6, all the one-loop results are collected to write the kernel W . And the renormalization group behavior of the hard function, the kernel and the PDF is discussed. A conclusion is presented in Sec. 7. In appendix A, the computation of the soft function at one loop with the nonzero offshellness of the external particles as the IR regulators is presented, and the collinear quark distribution functions and the quark PDF in the same regularization method is presented in appendix B. And it is also shown that the IR divergence and the mixed divergence cancel in the kernel.

2 Factorization theorems in SCET

In full QCD, the idea of factorization is probed in detail [1] to separate the hard, collinear and soft parts. In describing the soft part, the eikonal approximation is employed to show that it is decoupled from the collinear part. In SCET, the procedure of the factorization is elaborated since the factorization is achieved at the operator level. The collinear and soft parts are expressed in terms of the matrix elements of the operators, and they do not interact with each other from the outset by introducing the collinear and soft fields without any interaction between them. The scattering cross sections near the endpoint are

schematically written as

$$d\sigma = \begin{cases} H_{\text{DY}}(Q, \mu) \otimes S_{\text{DY}}(E, \mu) \otimes f_{q/N_1}(E, \mu) \otimes f_{\bar{q}/N_2}(E, \mu), \\ H_{\text{DIS}}(Q, \mu) \otimes J(Q\sqrt{1-z}, \mu) \otimes S_{\text{DIS}}(E, \mu) \otimes f_{q/N}(E, \mu), \\ H_{\text{Higgs}}(Q, \mu) \otimes S_{\text{Higgs}}(E, \mu) \otimes f_{g/N_1}(E, \mu) \otimes f_{g/N_2}(E, \mu), \end{cases} \quad (2.1)$$

in DY [13], DIS [14, 15] and Higgs [16] production processes respectively. $H(Q, \mu)$ is the hard function, Q is the large scale, and μ is the renormalization scale. $S(E, \mu)$ is the soft function, which is defined in terms of soft Wilson lines. The functions $f_{q/N}(E, \mu)$ and $f_{g/N}(E, \mu)$ are the collinear quark and gluon distribution functions defined in terms of the gauge-invariant operators at the intermediate scale $E \sim Q(1-z) \ll Q$, where z is the Bjorken variable. We reserve the terminology “parton distribution functions” for the same matrix elements as those for $f_{q/N}$ and $f_{g/N}$, but evaluated at a much lower scale than E . The PDF will be denoted as $\phi_{q/N}$ and $\phi_{g/N}$ from now on. The final-state jet function J appears only in DIS, and describes the collinear final-state particles. The symbol \otimes means an appropriate convolution, and the formulae with the explicit convolution will be presented below.

Eq. (2.1) is the conventional factorization theorem. However, as they stand, the soft functions S_{DY} , S_{DIS} and S_{Higgs} contain IR and mixed divergence, so do the collinear functions f_{q/N_1} , $f_{\bar{q}/N_2}$ and $f_{g/N}$. The fact that each factorized part can be computed separately has a merit, but the existence of IR divergences hinders us from using the evolution of each part. It can be cured by reorganizing the collinear and soft parts in such a way that the IR divergence is absent except in the PDF. The procedure of reorganization is not arbitrary, but is based on the physical principle that consistent separation of the collinear and soft modes should be maintained at higher loops.

Factorization theorems involve disparate scales, and it is more transparent and convenient to employ SCET to see the physics clearly. There are two scales Q and $E \sim Q(1-z)$, and the effective theories can be constructed by integrating out large scales successively, and the matching between the theories can be performed systematically. The first effective theory SCET_I can be constructed from full QCD by integrating out the degrees of freedom of order Q . When we consider DY and DIS processes in SCET_I, there appears a back-to-back current which can be written in SCET at leading order as

$$\bar{q}\gamma^\mu q = C(Q, \mu) \bar{\chi}_{\bar{n}} Y_{\bar{n}}^\dagger \gamma_\perp^\mu Y_n \chi_n + \text{h.c.}, \quad (2.2)$$

where $\chi_n = W_n^\dagger \xi_n$ is the n -collinear fermion field in SCET_I and $\chi_{\bar{n}} = W_{\bar{n}}^\dagger \xi_{\bar{n}}$ is the \bar{n} -collinear fermion. The collinear Wilson line W_n is introduced to make the current collinear gauge invariant and is given by

$$W_n = \sum_{\text{perm}} \exp \left[-\frac{g}{\bar{n} \cdot \mathcal{P} + i0} \bar{n} \cdot A_n \right], \quad (2.3)$$

where n and \bar{n} are lightcone vectors satisfying $n^2 = \bar{n}^2 = 0$, $n \cdot \bar{n} = 2$. The operator $\bar{n} \cdot \mathcal{P}$ extracts the label momentum, and the bracket means the operator acts only inside the bracket. The \bar{n} -collinear Wilson line $W_{\bar{n}}$ is obtained by switching n and \bar{n} in Eq. (2.3).

In the Higgs production, a back-to-back current with gluons is involved, and the detailed analysis is given in Sec. 2.3.

In addition to the UV and IR divergences, there may exist another type of divergence, called the rapidity divergence, which occurs when an energetic particle emits collinear gluons with infinite rapidity. When all the Feynman diagrams are added for a given quantity, the rapidity divergences cancel. But they appear in individual Feynman diagrams and a regularization is necessary to treat this divergence. It is suggested to include rapidity regulators to all the Wilson lines [17]. A simpler method to regulate the rapidity divergence is to insert rapidity regulators only in the collinear Wilson lines as [18]

$$W_n = \sum_{\text{perm}} \exp \left[-\frac{g}{\bar{n} \cdot \mathcal{P} + \delta_n + i0} \bar{n} \cdot A_n \right]. \quad (2.4)$$

The collinear Wilson line W_n can be heuristically derived by attaching n -collinear gluons to all the collinear or heavy particles not in the n direction, and integrating out the intermediate states. If these particles are on the mass shell, the gauge-invariant collinear Wilson line in Eq. (2.3) is obtained [19]. If those particles are slightly offshell, the collinear Wilson line takes the form in Eq. (2.4), where δ_n can be related to the combined offshellness of all the collinear and heavy particles not in the n direction. In this case, the gauge invariance of the collinear Wilson line is broken. But as long as the sum of all the diagrams is gauge-invariant, the introduction of the offshellness can be regarded just as an intermediate step. On the other hand, the regulator δ_n can be regarded simply as a regulator with no relation to the offshellness. The gauge-invariant formulation with the regulator can be probed, but we use Eq. (2.4) as it is to regulate the rapidity divergence. We will show the dependence on the rapidity regulator and its cancellation, when summed, at one loop below.

In Eq. (2.2), the collinear field ξ_n is redefined to be decoupled from the soft interactions as [5]

$$\xi_n \rightarrow Y_n \xi_n, A_n \rightarrow Y_n A_n Y_n^\dagger, \quad (2.5)$$

where the soft Wilson line Y_n is given by

$$Y_n = \sum_{\text{perm}} \exp \left[-\frac{g}{n \cdot \mathcal{R} + i0} n \cdot A_s \right]. \quad (2.6)$$

Here $\mathcal{R}^\mu = i\partial^\mu$ is a derivative operator on the soft field, which extracts soft momentum. In Eq. (2.6), the soft Wilson lines are collectively written as Y_n , but they should be determined according to the kinematic situations of the scattering process. The detailed prescription for the soft Wilson lines is discussed in Ref. [20], and it will be used in defining the soft function.

The matching between QCD and SCET_I is performed by comparing the matrix elements of the same quantities at the scale Q . The Wilson coefficient $C(Q^2, \mu)$ in Eq. (2.2), for example, is the matching coefficient for the back-to-back current. Since the IR divergence is cancelled in the matching, the Wilson coefficients are free of IR divergence. Now we go further down below E , and the degrees of freedom above E are integrated out to produce the final effective theory, SCET_{II}. The matching between SCET_I and SCET_{II} can also be performed systematically.

The key ingredient in adopting the successive effective theories is to relate the collinear quark and gluon distribution functions $f_{q/N}$ and $f_{g/N}$ in SCET_I to the quark and gluon PDF $\phi_{q/N}$ and $\phi_{g/N}$ in SCET_{II} as [22]

$$\begin{aligned} f_{q/N}(x, \mu) &= \int_x^1 \frac{dz}{z} K_{qq}(z, \mu) \phi_{q/N}(x/z, \mu), \\ f_{g/N}(x, \mu) &= \int_x^1 \frac{dz}{z} K_{gg}(z, \mu) \phi_{g/N}(x/z, \mu). \end{aligned} \quad (2.7)$$

We distinguish the collinear distribution functions $f_{q/N}$ and $f_{g/N}$ from the standard PDF $\phi_{q/N}$ and $\phi_{g/N}$ though the operator definitions for both are the same, but because they are evaluated at different energy scales. The collinear distribution functions are evaluated at an intermediate energy scale E ($\Lambda_{\text{QCD}} \ll E \ll Q$), while the PDF are evaluated at the scale much below E , but much larger than Λ_{QCD} . As will be shown below, this distinction is important. We call K_{qq} and K_{gg} the initial-state quark and gluon jet functions respectively. The initial-state jet function looks like the matching coefficient for the collinear matrix elements between SCET_I and SCET_{II}, but strictly speaking it is not. It includes IR divergences unlike the Wilson coefficient $C(Q^2, \mu)$. The initial-state jet function is actually the difference in the contributions of the soft modes in the two effective theories.

The soft function in SCET_I describes the contributions of the soft momentum of order $E \sim Q(1-z)$, hence the loop momentum of order $E \sim Q(1-z)$ should be subtracted from the collinear part to avoid double counting through the zero-bin subtraction. In SCET_{II}, the corresponding zero-bin subtraction should be performed with the momentum much smaller than $Q(1-z)$. This small momentum is sometimes referred to as the ultrasoft (usoft) momentum. These two kinds of zero-bin subtractions are different, and their difference yields the initial-state jet function. If there is no distinction between SCET_I and SCET_{II}, for example, away from the endpoint, the collinear distribution function and the PDF are identical and $K_{qq}(z, \mu) = K_{gg}(z, \mu) = \delta(1-z)$ to all orders in α_s , and there are no soft functions.

The initial-state jet function contains IR divergence, so does the soft function. However, the sum of these two functions is free of IR divergence. Furthermore the mixing of the UV and IR divergences is removed only in this sum. Therefore only the sum is physically meaningful and constitutes a component in the factorization theorem. What we propose as the proper factorization theorem is to use Eq. (2.7) to express the conventional factorization theorem Eq. (2.1) as

$$\begin{aligned} d\sigma_{\text{DY}} &= H_{\text{DY}}(Q, \mu) \otimes S_{\text{DY}}(E, \mu) \otimes K_{qq}(E, \mu) \otimes K_{\bar{q}\bar{q}}(E, \mu) \otimes \phi_{q/N_1}(\mu) \otimes \phi_{\bar{q}/N_2}(\mu) \\ &= H_{\text{DY}}(Q, \mu) \otimes W_{\text{DY}}(E, \mu) \otimes \phi_{q/N_1}(\mu) \otimes \phi_{\bar{q}/N_2}(\mu), \\ d\sigma_{\text{DIS}} &= H_{\text{DIS}}(Q, \mu) \otimes J(Q\sqrt{1-z}, \mu) \otimes S_{\text{DIS}}(E, \mu) \otimes K_{qq}(E, \mu) \otimes \phi_{q/N}(\mu) \\ &= H_{\text{DIS}}(Q, \mu) \otimes J(Q\sqrt{1-z}, \mu) \otimes W_{\text{DIS}}(E, \mu) \otimes \phi_{q/N}(\mu), \\ d\sigma_{\text{Higgs}} &= H_{\text{Higgs}}(Q, \mu) \otimes S_{\text{Higgs}}(E, \mu) \otimes K_{gg}(E, \mu) \otimes K_{gg}(E, \mu) \otimes \phi_{g/N_1}(\mu) \otimes \phi_{g/N_2}(\mu) \\ &= H_{\text{Higgs}}(Q, \mu) \otimes W_{\text{Higgs}}(E, \mu) \otimes \phi_{g/N_1}(\mu) \otimes \phi_{g/N_2}(\mu), \end{aligned} \quad (2.8)$$

for DY, DIS and the Higgs production processes respectively. Here we define the soft kernels W as

$$\begin{aligned} W_{\text{DY}}(E, \mu) &= S_{\text{DY}}(E, \mu) \otimes K_{qq}(E, \mu) \otimes K_{\bar{q}\bar{q}}(E, \mu), \\ W_{\text{DIS}}(E, \mu) &= S_{\text{DIS}}(E, \mu) \otimes K_{qq}(E, \mu), \\ W_{\text{Higgs}}(E, \mu) &= S_{\text{Higgs}}(E, \mu) \otimes K_{gg}(E, \mu) \otimes K_{gg}(E, \mu). \end{aligned} \quad (2.9)$$

Though the initial-state jet function is included in the kernel W , we call this the soft kernel since the initial-state jet function consists of the soft limits of the collinear parts, as will be explained below. Eq. (2.8) is our new factorization theorem, in which each factorized part can be systematically computed in perturbation theory. Furthermore, all the factors except the PDF are IR finite [22]. The explicit convoluted form and the one-loop results are shown in this paper.

One may wonder why we go through this labyrinthine procedure, while we know that the PDF in full QCD can be obtained by a straightforward calculation without the problem of IR divergence. It will be shown explicitly how full QCD works later. But to put it simply, the answer lies in the fact that full QCD computations correspond to the calculation of the PDF in SCET_{II}. In SCET_{II}, the usoft zero-bin contribution simply vanishes, hence no effect results from the usoft zero-bin subtraction. The details will be explained in Sec 4.

2.1 Drell-Yan process near the endpoint

Let us consider the inclusive Drell-Yan process $p(P_1)\bar{p}(P_2) \rightarrow l^+l^-(q) + X(p_X)$, where l^+l^- are a lepton pair and X denotes hadrons in the final state. We define the structure function $F_{\text{DY}}(\tau)$ as

$$F_{\text{DY}}(\tau) = -N_c \int \frac{d^4q}{(2\pi)^4} \theta(q^0) \delta(q^2 - s\tau) \int d^4z e^{-iq \cdot z} \langle N_1 N_2 | J_\mu^\dagger(z) J^\mu(0) | N_1 N_2 \rangle, \quad (2.10)$$

with the number of colors N_c , and $\tau = Q^2/s$, where Q^2 is the invariant mass squared of the lepton pair, and s is the hadronic center-of-mass energy squared. Near the endpoint $\tau \rightarrow 1$, the final-state particles are either soft with the interaction, or n - and \bar{n} -collinear without the interaction. The differential scattering cross section is given as $d\sigma/d\tau = \sigma_0 F_{\text{DY}}(\tau)$, where $\sigma_0 = 4\pi\alpha^2 Q_f^2 / (3N_c Q^2)$ is the Born cross section for the quark flavor f with the electric charge Q_f .

Now we express $F_{\text{DY}}(\tau)$ in SCET by using the current in Eq. (2.2), and the final state $|X\rangle$ is decomposed as $|X\rangle = |X_n\rangle |X_{\bar{n}}\rangle |X_s\rangle$, the n -, \bar{n} -collinear and the soft states. The momentum q^μ of the lepton pair is given by $q = P_1 + P_2 - p_X$, where $P_{1,2}$ is the momentum of the hadron $N_{1,2}$ in the n and \bar{n} directions respectively. These momenta are given by $P_1^\mu = \bar{n} \cdot P_1 n^\mu / 2$, and $P_2^\mu = n \cdot P_2 \bar{n}^\mu / 2$ where $s = \bar{n} \cdot P_1 n \cdot P_2$. The momentum of the final-state particles p_X can also be decomposed as $p_X = p_{X_n} + p_{X_{\bar{n}}} + p_{X_s}$, and q can be written as

$$q = P_1 + P_2 - (p_{X_n} + p_{X_{\bar{n}}} + p_{X_s}) = (P_1 - p_{X_n}) + (P_2 - p_{X_{\bar{n}}}) - p_{X_s} = p_1 + p_2 - p_{X_s}, \quad (2.11)$$

where $p_{1,2}$ are the momenta of the incoming partons inside the hadrons $N_{1,2}$ respectively. From now on, we express Eq. (2.10) in terms of the partonic variables. First the argument in the delta function in the partonic center-of-mass frame can be written as

$$\begin{aligned} q^2 - s\tau &= q^2 - Q^2 = (p_1 + p_2)^2 - 2p_{X_s} \cdot (p_1 + p_2) - Q^2 \\ &= \hat{s} - 2\eta\hat{s}^{1/2} - Q^2 \sim \hat{s}\left(1 - z - \frac{2\eta}{Q}\right), \end{aligned} \quad (2.12)$$

where $\eta = v \cdot p_{X_s} = (n \cdot p_{X_s} + \bar{n} \cdot p_{X_s})/2 = p_{X_s}^0$, and $z = Q^2/\hat{s}$, with the center-of-mass energy squared \hat{s} for the partons. We neglected $p_{X_s}^2$ in the first line. Near the endpoint, $\tau < z < 1$, $\tau \rightarrow 1$, and higher powers of $1 - z$ are neglected.

The structure function can be written in SCET as

$$\begin{aligned} F_{DY}(\tau) &= -\frac{N_c}{\hat{s}} H_{DY}(Q, \mu) \langle N_1 N_2 | \bar{\chi}_n Y_n^\dagger \gamma_\mu^\perp Y_{\bar{n}} \chi_{\bar{n}} \bar{\chi}_{\bar{n}} \delta\left(1 - z + \frac{2v \cdot \mathcal{R}}{Q}\right) Y_{\bar{n}}^\dagger Y_n \gamma_\mu^\perp \chi_n | N_1 N_2 \rangle \\ &= -N_c H_{DY}(Q, \mu) \int \frac{dy_1 dy_2}{\hat{s}} \langle N_1 N_2 | \bar{\chi}_n Y_n^\dagger \gamma_\mu^\perp Y_{\bar{n}} \chi_{\bar{n}} \\ &\quad \times \bar{\chi}_{\bar{n}} \delta\left(y_2 + \frac{n \cdot \mathcal{P}^\dagger}{n \cdot P_2}\right) \delta\left(1 - z + \frac{2v \cdot \mathcal{R}}{Q}\right) Y_{\bar{n}}^\dagger Y_n \gamma_\mu^\perp \delta\left(y_1 - \frac{\bar{n} \cdot \mathcal{P}}{\bar{n} \cdot P_1}\right) \chi_n | N_1 N_2 \rangle, \end{aligned} \quad (2.13)$$

where $H_{DY}(Q^\mu) = |C_{DY}(Q, \mu)|^2$ is the hard function and $\hat{s} = y_1 y_2 s$. From the last expression in Eq. (2.13), the soft interactions are decoupled, and the n - and \bar{n} -collinear parts are also decoupled since they no longer communicate to each other in SCET. The collinear matrix elements, after the operators are Fierz transformed and decoupled, can be written as

$$\begin{aligned} \langle N_1 | \left[\delta\left(y_1 - \frac{\bar{n} \cdot \mathcal{P}}{\bar{n} \cdot P_1}\right) \chi_n \right]_a^\alpha \left[\bar{\chi}_n \right]_b^\beta | N_1 \rangle &= \frac{\bar{n} \cdot P_1}{2N_c} \delta^{\alpha\beta} \left(\frac{\not{n}}{2}\right)_{ab} f_{q/N_1}(y_1), \\ \langle N_2 | \left[\chi_{\bar{n}} \right]_a^\alpha \left[\bar{\chi}_{\bar{n}} \delta\left(y_2 + \frac{n \cdot \mathcal{P}^\dagger}{n \cdot P_2}\right) \right]_b^\beta | N_2 \rangle &= \frac{n \cdot P_2}{2N_c} \delta^{\alpha\beta} \left(\frac{\not{\bar{n}}}{2}\right)_{ab} f_{\bar{q}/N_2}(y_2), \end{aligned} \quad (2.14)$$

where α, β are color indices, and a, b are Dirac indices. Eq. (2.14) defines the collinear quark and antiquark distribution functions for the incoming partons as

$$\begin{aligned} f_{q/N_1}(x_1) &= \langle N_1 | \bar{\chi}_n \frac{\not{n}}{2} \delta(x_1 \bar{n} \cdot P_1 - \bar{n} \cdot \mathcal{P}) \chi_n | N_1 \rangle, \\ f_{\bar{q}/N_2}(x_2) &= \langle N_2 | \bar{\chi}_{\bar{n}} \frac{\not{\bar{n}}}{2} \delta(x_2 n \cdot P_2 + n \cdot \mathcal{P}^\dagger) \chi_{\bar{n}} | N_2 \rangle. \end{aligned} \quad (2.15)$$

These collinear distribution functions are defined in SCET_I, that is, the matrix elements are evaluated at the scale above $E = Q(1 - z)$.

Finally the factorized form for $F_{DY}(\tau)$ is written as

$$\begin{aligned} F_{DY}(\tau) &= H_{DY}(Q, \mu) \int \frac{dy_1}{y_1} \frac{dy_2}{y_2} f_{q/N_1}(y_1) f_{\bar{q}/N_2}(y_2) S_{DY}(z, \mu) \\ &= H_{DY}(Q, \mu) \int_\tau^1 \frac{dz}{z} S_{DY}(z, \mu) F_{q\bar{q}}\left(\frac{\tau}{z}\right), \end{aligned} \quad (2.16)$$

where the soft function $S_{DY}(z, \mu)$ is defined as

$$S_{DY}(z, \mu) = \frac{1}{N_c} \langle 0 | \text{tr} Y_n^\dagger Y_{\bar{n}} \delta\left(1 - z + \frac{2v \cdot \mathcal{R}}{Q}\right) Y_{\bar{n}}^\dagger Y_n | 0 \rangle, \quad (2.17)$$

and is normalized to $\delta(1-z)$ at tree level. The soft Wilson lines $Y_{\bar{n}}^\dagger$ and Y_n are chosen such that the antiquark and the quark come from $-\infty$ [20], and their hermitian conjugates are employed in the left-hand side of the delta function. The function $F_{q\bar{q}}$ is given as

$$F_{q\bar{q}}\left(\frac{\tau}{z}\right) = \int_{\tau/z}^1 \frac{dy_1}{y_1} f_{q/N_1}(y_1) f_{\bar{q}/N_2}\left(\frac{\tau}{zy_1}\right). \quad (2.18)$$

Eq. (2.16) is the result of the conventional factorization.

In our factorization scheme, we further express the collinear function in terms of the PDF as

$$\begin{aligned} f_{q/N}(x, \mu) &= \int_x^1 \frac{dz}{z} K_{qq}(z, \mu) \phi_{q/N}(x/z, \mu), \\ f_{\bar{q}/N}(x, \mu) &= \int_x^1 \frac{dz}{z} K_{\bar{q}\bar{q}}(z, \mu) \phi_{\bar{q}/N}(x/z, \mu). \end{aligned} \quad (2.19)$$

By shuffling the order of integrations, and changing the integration variables, F_{DY} is written as

$$\begin{aligned} F_{\text{DY}}(\tau) &= H_{\text{DY}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} \int_{\tau/w}^1 \frac{dx}{x} \phi_{q/N_1}(x, \mu) \phi_{\bar{q}/N_2}\left(\frac{\tau}{xw}, \mu\right) \\ &\quad \times \int_w^1 \frac{dz}{z} S_{\text{DY}}(z, \mu) \int_{w/z}^1 \frac{dt}{t} K_{qq}(t, \mu) K_{\bar{q}\bar{q}}\left(\frac{w}{zt}, \mu\right) \\ &= H_{\text{DY}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} W_{\text{DY}}(w, \mu) \int_{\tau/w}^1 \frac{dx}{x} \phi_{q/N_1}(x, \mu) \phi_{\bar{q}/N_2}\left(\frac{\tau}{xw}, \mu\right) \\ &= H_{\text{DY}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} W_{\text{DY}}(w, \mu) \Phi_{q\bar{q}}\left(\frac{\tau}{w}, \mu\right), \end{aligned} \quad (2.20)$$

where W_{DY} and $\Phi_{q\bar{q}}$ are given as

$$\begin{aligned} W_{\text{DY}}(w, \mu) &= \int_w^1 \frac{dz}{z} S_{\text{DY}}(z, \mu) \int_{w/z}^1 \frac{dt}{t} K_{qq}(t, \mu) K_{\bar{q}\bar{q}}\left(\frac{w}{zt}, \mu\right), \\ \Phi_{q\bar{q}}\left(\frac{\tau}{w}, \mu\right) &= \int_{\tau/w}^1 \frac{dx}{x} \phi_{q/N_1}(x, \mu) \phi_{\bar{q}/N_2}\left(\frac{\tau}{xw}, \mu\right). \end{aligned} \quad (2.21)$$

Eq. (2.20) is our new factorization theorem for DY process near the endpoint. The structure function is written as the convolution of the hard function, the soft kernel W_{DY} , and the product of the PDFs. And it will be shown that W_{DY} is IR finite at one loop.

2.2 Deep inelastic scattering near the endpoint

Consider the inclusive deep inelastic scattering $e(k_1) + p(P) \rightarrow e(k_2) + X(p_X)$ near the endpoint. The hadronic tensor $W^{\mu\nu}$ is written as

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{2\pi} \sum_X \int d^4z e^{iq \cdot z} \langle N(P) | J^{\mu\dagger}(z) | X(p_X) \rangle \langle X(p_X) | J^\nu(0) | N(P) \rangle \\ &= \frac{1}{2\pi} \sum_X (2\pi)^4 \delta^{(4)}(q + P - p_X) \langle N | J^{\mu\dagger} | X \rangle \langle X | J^\nu | N \rangle, \end{aligned} \quad (2.22)$$

where $q = k_2 - k_1$ is the momentum transfer from the leptonic system. The summation \sum_X includes all the phase spaces of the final-state particles. Here we consider the electromagnetic current only, but the inclusion of the weak current is straightforward. In the Breit frame, the incoming proton is in the n direction, and the outgoing particles are in the \bar{n} direction. The analysis near the endpoint in DIS using SCET was first performed both in the Breit and the target-rest frames in Ref. [21].

The momentum q and the momentum P of the incoming proton can be written as

$$q^\mu = \frac{\bar{n}^\mu}{2}Q - \frac{n^\mu}{2}Q, \quad P^\mu = \bar{n} \cdot P \frac{n^\mu}{2}, \quad (2.23)$$

and the Bjorken variable is defined as

$$x = \frac{Q^2}{2P \cdot q} \sim \frac{Q}{\bar{n} \cdot P}. \quad (2.24)$$

The endpoint corresponds to the limit $x \sim 1$.

The hadronic tensor can be expressed in terms of the structure functions as [14]

$$W^{\mu\nu} = -g_\perp^{\mu\nu} F_1 + v^\mu v^\nu F_L, \quad (2.25)$$

with $v^\mu = (n^\mu + \bar{n}^\mu)/2$. The structure function F_1 is given by

$$\begin{aligned} F_1 &= -\frac{1}{2}g_\perp^{\mu\nu}W_{\mu\nu} = -\frac{1}{4\pi} \sum_X (2\pi)^4 \delta^{(4)}(q + P - p_X) \langle N | J^{\mu\dagger} | X \rangle \langle X | J_\mu | N \rangle \\ &= -\frac{1}{4\pi} \sum_X (2\pi)^4 \delta^{(4)}(q + P - p_X) |C_{\text{DIS}}(Q, \mu)|^2 \\ &\quad \times \int_0^1 dy \langle N | \bar{\chi}_n Y_n^\dagger \gamma_\perp^\mu \tilde{Y}_n \chi_{\bar{n}} | X \rangle \langle X | \bar{\chi}_{\bar{n}} \tilde{Y}_n^\dagger \gamma_\mu^\perp Y_n \delta\left(y - \frac{\bar{n} \cdot P}{\bar{n} \cdot P}\right) \chi_n | N \rangle, \end{aligned} \quad (2.26)$$

where the current J^μ in SCET is given by

$$J^\mu = C_{\text{DIS}}(Q, \mu) \bar{\chi}_{\bar{n}} \tilde{Y}_n^\dagger \gamma_\perp^\mu Y_n \chi_n. \quad (2.27)$$

Here the soft Wilson lines are chosen such that the incoming quark comes from $-\infty$, and the outgoing quark goes to ∞ .

We now define the two collinear functions in the \bar{n} and n directions, which do not communicate to each other. The final-state jet function in the \bar{n} direction is defined as

$$\sum_{X_{\bar{n}}} \chi_{\bar{n}} | X_{\bar{n}} \rangle \langle X_{\bar{n}} | \bar{\chi}_{\bar{n}} = \frac{\not{\bar{n}}}{2} \int \frac{d^4 p_{X_{\bar{n}}}}{(2\pi)^4} \bar{J}(p_{X_{\bar{n}}}). \quad (2.28)$$

The jet function is a function of $n \cdot p_{X_{\bar{n}}}$ only. The collinear distribution function in the n direction is defined, as in Eq. (2.14), by

$$\sum_{X_n} \langle N | (\bar{\chi}_n)_a^\alpha | X_n \rangle \langle X_n | \delta\left(y - \frac{\bar{n} \cdot P}{\bar{n} \cdot P}\right) (\chi_n)_b^\beta | N \rangle = \frac{\bar{n} \cdot P}{2N_c} \delta^{\alpha\beta} \left(\frac{\not{n}}{2}\right)_{ba} f_{q/N}(y). \quad (2.29)$$

In terms of these two collinear functions, the structure function F_1 can be written as

$$F_1(x) = |C_{\text{DIS}}(Q, \mu)|^2 \int_x^1 \frac{dy}{y} f_{q/N}(y, \mu) \times \int_0^{Q(1-x/y)} \frac{d\bar{n} \cdot k}{2\pi} \bar{J}(\bar{n} \cdot k, \mu) \bar{S}_{\text{DIS}}\left(1 - \frac{x}{y} - \frac{\bar{n} \cdot k}{Q}, \mu\right), \quad (2.30)$$

where the soft function \bar{S}_{DIS} is defined as

$$\bar{S}_{\text{DIS}}(1-z) = \frac{1}{N_c} \langle 0 | \text{tr} \left[Y_n^\dagger \tilde{Y}_{\bar{n}} \delta\left(1-z + \frac{\bar{n} \cdot \mathcal{R}}{Q}\right) \tilde{Y}_{\bar{n}}^\dagger Y_n \right] | 0 \rangle. \quad (2.31)$$

Changing the variables $\bar{n} \cdot k = (1-w)Q$, and defining

$$J(w, \mu) = \frac{Q}{2\pi} \bar{J}(Q(1-w), \mu), \quad S_{\text{DIS}}\left(\frac{z}{w}, \mu\right) = w \bar{S}_{\text{DIS}}(w-z), \quad (2.32)$$

The jet function $J(w, \mu)$ and the soft function $S_{\text{DIS}}(w, \mu)$ are normalized to $\delta(1-w)$ respectively at tree level. Eq. (2.30) is written as

$$F_1(x) = H_{\text{DIS}}(Q, \mu) \int_x^1 \frac{dz}{z} f_{q/N}\left(\frac{x}{z}, \mu\right) \int_z^1 \frac{dw}{w} J(w, \mu) S_{\text{DIS}}\left(\frac{z}{w}, \mu\right), \quad (2.33)$$

where $H_{\text{DIS}}(Q, \mu) = |C_{\text{DIS}}(Q, \mu)|^2$, and subleading terms of S_{DIS} in w is neglected near the endpoint.

Eq. (2.33) is the result of the conventional factorization theorem near the endpoint. In our formulation, as was done in DY process, we go further by invoking Eq. (2.19) to write Eq. (2.33) as

$$F_1(x) = H_{\text{DIS}}(Q, \mu) \int_x^1 \frac{dz}{z} \int_{x/z}^1 \frac{dy}{y} K_{qq}(y) \phi\left(\frac{x}{yz}, \mu\right) \int_z^1 \frac{dw}{w} J(w) S_{\text{DIS}}\left(\frac{z}{w}, \mu\right). \quad (2.34)$$

Changing the order of integrations successively, and redefining the integration variables, the factorized structure function is given as

$$\begin{aligned} F_1(x) &= H(Q, \mu) \int_x^1 \frac{dw}{w} J(w) \int_{x/w}^1 \frac{dv}{v} \left[\int_v^1 \frac{du}{u} S_{\text{DIS}}(u, \mu) K_{qq}\left(\frac{v}{u}, \mu\right) \right] \phi_{q/N}\left(\frac{x}{vw}, \mu\right) \\ &= H(Q, \mu) \int_x^1 \frac{dw}{w} J(w) \int_{x/w}^1 \frac{dv}{v} W_{\text{DIS}}(v, \mu) \phi_{q/N}\left(\frac{x}{vw}, \mu\right), \end{aligned} \quad (2.35)$$

where $W_{\text{DIS}}(v, \mu)$ is given by

$$W_{\text{DIS}}(v, \mu) = \int_v^1 \frac{du}{u} S_{\text{DIS}}(u, \mu) K_{qq}\left(\frac{v}{u}, \mu\right). \quad (2.36)$$

Eq. (2.35) is our new result of the factorization theorem in deep inelastic scattering. The soft kernel W_{DIS} is IR finite, and surprisingly enough its radiative corrections to all orders vanish and $W_{\text{DIS}}(v) = \delta(1-v)$. The argument about why it is true to all orders in α_s is given after the one-loop corrections are presented.

2.3 Higgs production near the partonic threshold

The factorization proof for the Higgs production is similar to that for DY process except that the initial-state particles are gluons instead of a $q\bar{q}$ pair. Therefore this process involves the gluon PDF and the initial-state gluon jet function. The effective Lagrangian for the Higgs production after integrating out the top quark loop is given by

$$\mathcal{L}_{\text{eff}} = C_t(m_t, \mu) \frac{H}{v} G_{\mu\nu}^a G^{\mu\nu a}, \quad (2.37)$$

where v is the electroweak vacuum expectation value, m_t is the top quark mass, H is the Higgs field and $G_{\mu\nu}^a$ is the field strength tensor for gluons. The coefficient $C_t(m_t^2, \mu)$ to second order in α_s , and in the heavy top quark mass limit, is given by [23, 24]

$$C_t(m_t, \mu) = \frac{\alpha_s(\mu)}{12\pi} \left(1 + \frac{\alpha_s(\mu)}{\pi} \frac{11}{4} \right). \quad (2.38)$$

The gluon field strength tensor is matched onto SCET as

$$G_{\mu\nu}^a G^{\mu\nu a} \rightarrow -2C_H(Q, \mu) \mathcal{B}_n^{\perp\mu} \mathcal{Y}_n^\dagger \mathcal{Y}_n \mathcal{B}_{n\perp}^\mu = -2C_H(Q, \mu) O_g, \quad (2.39)$$

after decoupling the soft interaction. In this process, $Q = m_H$, the Higgs mass. Here $\mathcal{B}_n^{\perp\mu}$ is defined as

$$\mathcal{B}_n^{\perp\mu} = \frac{1}{g} [\bar{n} \cdot \mathcal{P} W_n^\dagger i D_n^{\perp\mu} W_n] = i \bar{n}_\alpha g_{\perp\beta}^\mu W_n^\dagger G_n^{\alpha\beta} W_n = i \bar{n}_\alpha g_{\perp\beta}^\mu T^a (\mathcal{W}_n^\dagger)^{ab} G_n^{\alpha\beta, b}. \quad (2.40)$$

In the final expression, the collinear Wilson line \mathcal{W}_n is written in the adjoint representation rather than in the fundamental representation. That means that the generator T^a is given by $(T^a)^{bc} = -if^{abc}$. The collinear Wilson lines can be expressed either in the fundamental representation or in the adjoint representation. Both approaches are equivalent, but the adjoint representation is employed here since it makes the expression in the factorization formula look similar to that in DY processes. The Wilson coefficient C_H is the matching coefficient between the full theory and SCET and is given to order α_s by [25]

$$C_H(Q, \mu) = 1 + \frac{\alpha_s C_A}{4\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} + \frac{7}{6} \pi^2 - 2i\pi \ln \frac{\mu^2}{Q^2} \right), \quad (2.41)$$

where $C_A = N_c$. Then the effective Lagrangian in SCET is given by

$$\mathcal{L}_{\text{SCET}} = -\frac{2}{v} C_t(m_t, \mu) C_H(Q, \mu) H O_g = -C(\mu) H O_g. \quad (2.42)$$

The kinematics of the Higgs production process $p(P_1)p(P_2) \rightarrow HX$ is similar to that in DY process. The momentum q of the Higgs particle is given by

$$q = P_1 + P_2 - p_X = p_1 + p_2 - p_{X_s}, \quad (2.43)$$

as in Eq. (2.11). The cross section for the Higgs production process is written as

$$\begin{aligned} \sigma(pp \rightarrow HX) &= \frac{\pi}{s} \sum_X \int d^4q \delta^{(4)}(P_1 + P_2 - q - p_X) \delta(q^2 - Q^2) \theta(q^0) \\ &\quad \times |C(\mu)|^2 \langle N_1 N_2 | O_g^\dagger | X \rangle \langle X | O_g | N_1 N_2 \rangle \\ &= \frac{\pi}{s} |C(\mu)|^2 \int dx_1 dx_2 \langle N_1 N_2 | \mathcal{B}_n^{\perp\mu a} \mathcal{Y}_n^{\dagger ab} \mathcal{Y}_n^{bc} \mathcal{B}_{\bar{n}\mu}^{\perp c} \frac{1}{\hat{s}} \delta \left(1 - z - \frac{2\eta}{Q} \right) \\ &\quad \times \left[\delta \left(x_2 - \frac{n \cdot \mathcal{P}}{n \cdot P_2} \right) \mathcal{B}_n^{\perp\mu d} \right] \mathcal{Y}_n^{\dagger de} \mathcal{Y}_n^{ef} \left[\delta \left(x_1 - \frac{\bar{n} \cdot \mathcal{P}}{\bar{n} \cdot P_1} \right) \mathcal{B}_n^{\perp\mu f} \right] | N_1 N_2 \rangle, \end{aligned} \quad (2.44)$$

where $\eta = v \cdot p_X = p_X^0$ and $z = Q^2/\hat{s}$ ($\tau = Q^2/s$) with the center-of-mass energy squared for the partons (the hadrons). The partonic threshold corresponds to the limit $z \rightarrow 1$. As in DY process, the soft interactions are decoupled, so are the n - and \bar{n} -collinear parts. The collinear matrix elements are written as

$$\langle N_1 | \mathcal{B}_n^{\perp \mu a} \left[\delta \left(x_1 - \frac{\bar{n} \cdot \mathcal{P}}{\bar{n} \cdot P_1} \right) \mathcal{B}_n^{\perp \nu f} \right] | N_1 \rangle = \frac{g_{\perp}^{\mu \nu} \delta^{af}}{2(N_c^2 - 1)} x (\bar{n} \cdot P_1)^2 f_{g/N_1}(x). \quad (2.45)$$

Inverting Eq. (2.45), the collinear gluon distribution amplitude f_{g/N_1} is written as

$$f_{g/N_1}(x) = \frac{1}{x(\bar{n} \cdot P_1)^2} \langle N_1 | \mathcal{B}_n^{\perp \mu a} \left[\delta \left(x_1 - \frac{\bar{n} \cdot \mathcal{P}}{\bar{n} \cdot P_1} \right) \mathcal{B}_{n\mu}^{\perp a} \right] | N_1 \rangle. \quad (2.46)$$

For f_{g/N_2} , n and \bar{n} are switched in Eq. (2.46).

In terms of the collinear gluon distribution functions, the cross section is written as

$$\sigma(pp \rightarrow HX) = \sigma_0 \int dx_1 dx_2 H_{\text{Higgs}}(Q, \mu) S_{\text{Higgs}}(z, \mu) f_{g/N_1}(x_1, \mu) f_{g/N_2}(x_2, \mu), \quad (2.47)$$

where the soft function $S_{\text{Higgs}}(z, \mu)$ is defined as

$$S_{\text{Higgs}}(z, \mu) = \frac{1}{N_c^2 - 1} \langle 0 | \text{tr} \mathcal{Y}_n^\dagger \mathcal{Y}_{\bar{n}} \delta \left(1 - z + \frac{2v \cdot \mathcal{R}}{Q} \right) \mathcal{Y}_{\bar{n}}^\dagger \mathcal{Y}_n | 0 \rangle. \quad (2.48)$$

The Born cross section σ_0 is given by

$$\sigma_0 = \frac{2\pi |C_t(m_t, \mu)|^2}{v^2(N_c^2 - 1)}, \quad (2.49)$$

and $H_{\text{Higgs}}(Q, \mu) = |C_H(Q, \mu)|^2$.

Here we invoke the relation Eq. (2.7) to write the cross section as

$$\begin{aligned} \sigma(pp \rightarrow HX)/\sigma_0 &= H_{\text{Higgs}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} \int_{\tau/w}^1 dx \phi_{g/N_1}(x, \mu) \phi_{g/N_2}\left(\frac{\tau}{xw}, \mu\right) \\ &\quad \times \int_w^1 \frac{dz}{z} S_{\text{Higgs}}(z, \mu) \int_{w/z}^1 dt K_{gg}(t, \mu) K_{gg}\left(\frac{w}{zt}, \mu\right) \\ &= H_{\text{Higgs}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} W_{\text{Higgs}}(w, \mu) \int_{\tau/w}^1 dx \phi_{g/N_1}(x, \mu) \phi_{g/N_2}\left(\frac{\tau}{xw}, \mu\right) \\ &= H_{\text{Higgs}}(Q, \mu) \int_{\tau}^1 \frac{dw}{w} W_{\text{Higgs}}(w, \mu) \Phi_{gg}\left(\frac{\tau}{w}, \mu\right), \end{aligned} \quad (2.50)$$

where W_{Higgs} and Φ_{gg} are given as

$$\begin{aligned} W_{\text{Higgs}}(w, \mu) &= \int_w^1 \frac{dz}{z} S_{\text{Higgs}}(z, \mu) \int_{w/z}^1 dt K_{gg}(t, \mu) K_{gg}\left(\frac{w}{zt}, \mu\right), \\ \Phi_{gg}\left(\frac{\tau}{w}, \mu\right) &= \int_{\tau/w}^1 dx \phi_{g/N_1}(x, \mu) \phi_{g/N_2}\left(\frac{\tau}{xw}, \mu\right). \end{aligned} \quad (2.51)$$

Eq. (2.50) is our new factorization theorem for the Higgs production near the partonic threshold.

3 Soft function

In the conventional factorization scheme, the most problematic part is the soft function because it includes the IR and mixed divergences. Unless these divergences are removed, the soft function is not physical and the evolution via the renormalization group equation is meaningless. Especially the real gluon emission in DY process has only IR divergences. From this section, the ingredients in the factorization formulae Eqs. (2.20), and (2.35) are computed at one loop. The UV and IR divergences appear unequivocally in the soft functions, and we compute the one-loop corrections to the soft functions.

In DY process, the soft function is given by

$$S_{\text{DY}}(z, \mu) = \frac{1}{N_c} \langle 0 | \text{tr} Y_n^\dagger Y_{\bar{n}} \delta\left(1 - z + \frac{2v \cdot \mathcal{R}}{Q}\right) Y_{\bar{n}}^\dagger Y_n | 0 \rangle, \quad (3.1)$$

The Feynman diagrams for the radiative corrections of the soft function at one loop are given in Fig. 1. Fig. 1 (a) with its mirror image corresponds to the virtual correction, and Fig. 1(b) is the real gluon emission. The computation is performed by employing the dimensional regularization to regulate both the UV and IR divergences with the spacetime dimension $D = 4 - 2\epsilon$ and the $\overline{\text{MS}}$ scheme. The results with another regularization scheme, in which the dimensional regularization is employed for the UV divergence, and the IR divergence is regulated by the offshellness, are presented in Appendix A.

The corresponding matrix elements are given as

$$\begin{aligned} M_{s,\text{DY}}^a &= -2ig^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 n \cdot l \bar{n} \cdot l} = -\frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right)^2 \delta(1 - z), \\ M_{s,\text{DY}}^b &= 4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{n \cdot l \bar{n} \cdot l} \delta(l^2) \delta\left(1 - z - \frac{2v \cdot l}{Q}\right) \\ &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1 - z) \left(\frac{1}{\epsilon_{\text{IR}}^2} + \frac{2}{\epsilon_{\text{IR}}} \ln \frac{\mu}{Q} + 2 \ln^2 \frac{\mu}{Q} - \frac{\pi^2}{4} \right) \right. \\ &\quad \left. - \frac{2}{(1 - z)_+} \left(\frac{1}{\epsilon_{\text{IR}}} + 2 \ln \frac{\mu}{Q} \right) + 4 \left(\frac{\ln(1 - z)}{1 - z} \right)_+ \right]. \end{aligned} \quad (3.2)$$

In this computation, the function $1/(1 - z)^{1+\epsilon}$ appears. It diverges at $z = 1$, and the divergence is definitely of the IR origin. In terms of the plus distribution functions, it can

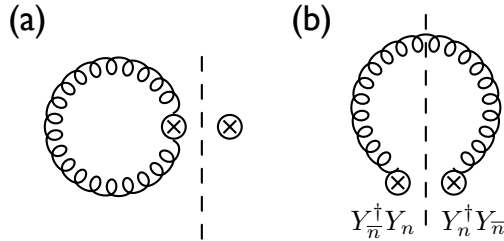


Figure 1. Feynman diagrams for soft functions at one loop (a) virtual corrections and (b) real gluon emission.

be expanded in powers of ϵ as

$$\frac{1}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(1-z) + \frac{1}{(1-z)_+} - \epsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + \dots \quad (3.3)$$

Here $M_{s,\text{DY}}^b$ is the result with Y_n ($Y_{\bar{n}}$) in the left- (right-) hand side in Fig. 1 (b), and the result with $Y_{\bar{n}}^\dagger$ and Y_n^\dagger should be included. They are hermitian conjugates to each other. The virtual correction has the UV and IR divergences, and the mixed divergence. Note that there are only IR divergences in the real gluon emission.

The total soft contribution for DY process at one loop is given as

$$\begin{aligned} S_{\text{DY}}^{(1)}(z) &= 2\text{Re}(M_{s,\text{DY}}^a + M_{s,\text{DY}}^b) \\ &= \frac{\alpha_s C_F}{\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon_{\text{UV}}^2} + \frac{2}{\epsilon_{\text{UV}}\epsilon_{\text{IR}}} + \frac{1}{\epsilon_{\text{IR}}} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{4} \right) \right. \\ &\quad \left. - \frac{2}{(1-z)_+} \left(\frac{1}{\epsilon_{\text{IR}}} + \ln \frac{\mu^2}{Q^2} \right) + 4 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right]. \end{aligned} \quad (3.4)$$

In Eq. (3.4), the finite terms are the same as the result in Ref. [26]. But the divergent terms are mixture of IR and UV divergences contrary to the argument in Ref. [26], in which they claim that the IR divergences are cancelled due to the KLN theorem. Our explicit calculation shows that it is not true. The KLN theorem holds when there is no kinematic constraint on the real gluon emission. However, as can be seen in $M_{s,\text{DY}}^b$ for the real gluon emission, there is a constraint for the soft momentum of the real gluon specified by the delta function, which is different from the virtual corrections. Therefore the cancellation of the IR divergence is bound to be incomplete.

The soft function for the Higgs production $S_{\text{Higgs}}(z)$ is defined as

$$S_{\text{Higgs}}(z) = \frac{1}{N_c^2 - 1} \langle 0 | \text{tr} \mathcal{Y}_n^\dagger \mathcal{Y}_{\bar{n}} \delta\left(1-z + \frac{2v \cdot \mathcal{R}}{Q}\right) \mathcal{Y}_{\bar{n}}^\dagger \mathcal{Y}_n | 0 \rangle, \quad (3.5)$$

where $Q = m_H$, and the soft Wilson lines \mathcal{Y}_n , and $\mathcal{Y}_{\bar{n}}$ are in the adjoint representation. $S_{\text{Higgs}}(z)$ is normalized to $\delta(1-z)$ at tree level. In this expression, Eq. (3.5) resembles Eq. (3.1), and the only difference is the representation of the soft Wilson lines. This means that the radiative corrections for S_{Higgs} is the same as S_{DY} except the color factors. By explicit computation, it is true and the corresponding radiative corrections for S_{Higgs} are obtained from Eq. (3.2) by replacing C_F by C_A . Accordingly, the one-loop correction for S_{Higgs} is given by

$$\begin{aligned} S_{\text{Higgs}}^{(1)}(z) &= \frac{\alpha_s C_A}{\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon_{\text{UV}}^2} + \frac{2}{\epsilon_{\text{UV}}\epsilon_{\text{IR}}} + \frac{1}{\epsilon_{\text{IR}}} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{4} \right) \right. \\ &\quad \left. - \frac{2}{(1-z)_+} \left(\frac{1}{\epsilon_{\text{IR}}} + \ln \frac{\mu^2}{Q^2} \right) + 4 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right]. \end{aligned} \quad (3.6)$$

The soft function S_{DIS} in DIS near the endpoint is given by

$$S_{\text{DIS}}(z) = \frac{1}{N_c} \langle 0 | \text{tr} \left[Y_n^\dagger \tilde{Y}_{\bar{n}} \delta\left(1-z + \frac{\bar{n} \cdot \mathcal{R}}{Q}\right) \tilde{Y}_{\bar{n}}^\dagger Y_n \right] | 0 \rangle. \quad (3.7)$$

The Feynman diagrams for the one-loop correction is shown in Fig. 1, except that $Y_{\bar{n}}$ is replaced by $\tilde{Y}_{\bar{n}}$. The matrix elements are given as

$$\begin{aligned} M_{s,\text{DIS}}^a &= -\frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right)^2, \\ M_{s,\text{DIS}}^b &= -\frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(\frac{1}{\epsilon_{\text{IR}}} + \frac{1}{2} \ln \frac{\mu^2}{Q^2} \right) \delta(1-z) - \frac{1}{(1-z)_+} \right]. \end{aligned} \quad (3.8)$$

Unlike DY process, the real gluon emission contains UV divergence. The total soft contribution in DIS at one loop is given by

$$\begin{aligned} S_{\text{DIS}}^{(1)}(z) &= 2\text{Re}(M_{s,\text{DIS}}^a + M_{s,\text{DIS}}^b) \\ &= \frac{\alpha_s C_F}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(-\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{2} \ln \frac{\mu^2}{Q^2} \right) \delta(1-z) + \frac{1}{(1-z)_+} \right]. \end{aligned} \quad (3.9)$$

All the soft functions contain IR, UV divergences as well as mixed divergence. Therefore they are not physically meaningful as they are. As claimed, only after the initial-state jet function is added, the IR and mixed divergences disappear.

4 Collinear distribution functions and PDF

4.1 Collinear quark distribution function and PDF

The radiative corrections of the collinear quark distribution function, defined as

$$f_{q/N}(x, \mu) = \langle N | \bar{\chi}_n \frac{\not{p}}{2} \delta(\bar{n} \cdot Px - \bar{n} \cdot \mathcal{P}) \chi_n | N \rangle, \quad (4.1)$$

will be computed explicitly at one loop. The Feynman diagrams for the collinear function at one loop are shown in Fig. 2. Fig. 2 (a) is the virtual correction, and Fig. 2 (b) and (c) are real gluon emissions. The mirror images of Fig. 2 (a) and (b) are omitted, which are given by the hermitian conjugates of the original diagrams.

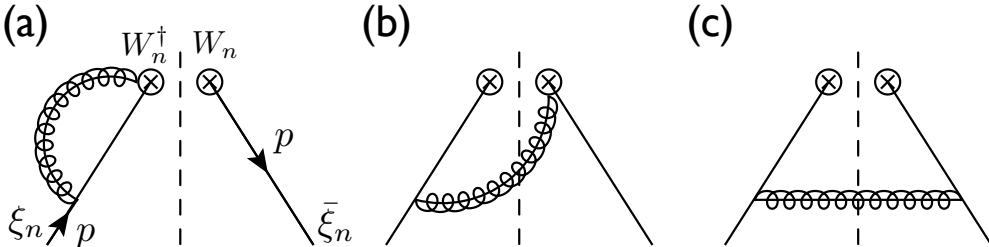


Figure 2. Feynman diagrams for collinear functions and PDF at one loop (a) virtual corrections, (b) and (c) real gluon emission. The mirror images of (a) and (b) are omitted.

The matrix elements are written as

$$\begin{aligned}
M_a &= 2ig^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \delta(1-x) \int \frac{d^D l}{(2\pi)^D} \frac{\bar{n} \cdot (p-l)}{l^2 (l-p)^2 (\bar{n} \cdot l + \delta_1)}, \\
M_b &= -4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{\bar{n} \cdot (p-l)}{(l-p)^2 (\bar{n} \cdot l + \delta_1)} \delta\left(1-x - \frac{\bar{n} \cdot l}{\bar{n} \cdot p}\right) \delta(l^2), \\
M_c &= 2\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon (D-2) \int \frac{d^D l}{(2\pi)^D} \frac{\mathbf{l}_\perp^2}{[(l-p)^2]^2} \delta\left(1-x - \frac{\bar{n} \cdot l}{\bar{n} \cdot p}\right) \delta(l^2). \quad (4.2)
\end{aligned}$$

These are computed using the dimensional regularization for both UV and IR divergences with $p^2 = 0$ in the $\overline{\text{MS}}$ scheme. The results with the dimensional regularization for the UV divergence, and the nonzero p^2 for the IR divergence are presented in Appendix B.

The results of the computation are given by

$$\begin{aligned}
M_a &= \frac{\alpha_s C_F}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(1 + \ln \frac{\delta_1}{\bar{n} \cdot p} \right), \\
M_b &= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(-\delta(1-x) \ln \frac{\delta_1}{\bar{n} \cdot p} + \frac{x}{(1-x)_+} \right), \\
M_c &= \frac{\alpha_s C_F}{2\pi} (1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right). \quad (4.3)
\end{aligned}$$

The rapidity regulator δ_1 is employed in the computation. As can be seen, M_a and M_b depend on this regulator δ_1 , but this dependence is cancelled in the sum $M_a + M_b$ even without the zero-bin subtraction. This is to be contrasted with the transverse-momentum-dependent collinear function, where the cancellation of δ_1 is achieved only after the zero-bin subtraction [18].

In computing the zero-bin contributions, we neglect all the components of the loop momentum l compared to $\bar{n} \cdot p$. From Eq. (4.2), they are given as

$$\begin{aligned}
M_a^{(0)} &= -2ig^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \delta(1-x) \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 n \cdot l (\bar{n} \cdot l + \delta_1)} \\
&= -\frac{\alpha_s C_F}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(\frac{1}{\epsilon_{\text{UV}}} - \ln \frac{\delta_1}{\mu} \right), \\
M_{b,s}^{(0)} &= 4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{n \cdot l (\bar{n} \cdot l + \delta_1)} \delta\left(1-x - \frac{\bar{n} \cdot l}{\bar{n} \cdot p}\right) \delta(l^2) \\
&= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(-\delta(1-x) \ln \frac{\delta_1}{\bar{n} \cdot p} + \frac{1}{(1-x)_+} \right), \\
M_c^{(0)} &= 0. \quad (4.4)
\end{aligned}$$

The zero-bin contribution $M_c^{(0)}$ is subleading and is neglected. Note that the distinction between the soft and usoft zero-bin contributions appears in $M_{b,s}^{(0)}$. In SCET_I, the soft momentum is of order $Q(1-x)$, which is of the same order as the loop momentum $\bar{n} \cdot l$ in the zero-bin contribution. However, there is no distinction between the soft and usoft contributions in $M_a^{(0)}$ because the integral remains the same irrespective of the size of the loop momentum. $M_c^{(0)}$ can also be neglected in the usoft limit. Therefore the soft and usoft zero-bin contributions are different only in M_b , which corresponds to the real gluon emission with soft or usoft momentum.

The collinear part with the soft zero-bin subtraction is written as

$$\begin{aligned}\tilde{M}_a &= M_a - M_a^{(0)} = \frac{\alpha_s C_F}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(\frac{1}{\epsilon_{\text{UV}}} + 1 + \ln \frac{\mu}{\bar{n} \cdot p} \right), \\ \tilde{M}_b &= M_b - M_{b,s}^{(0)} = -\frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right), \\ \tilde{M}_c &= M_c = \frac{\alpha_s C_F}{2\pi} (1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right).\end{aligned}\tag{4.5}$$

Note that various combinations are independent of the rapidity regulator δ_1 though individual diagrams depend on it. As was first noted, the sum of naive collinear contributions is independent of δ_1 . So is the sum of soft zero-bin contributions, hence the true collinear contribution with the zero-bin subtraction. It is also true that the usoft zero-bin contribution is also independent of δ_1 , in fact, it vanishes. It turns out that the same result is obtained without introducing δ_1 in the beginning. But δ_1 is included in the calculation to show how the cancellation occurs explicitly, and this method has been used also in calculating the transverse-momentum-dependent collinear distribution function [18].

The collinear distribution function at one loop is given by

$$\begin{aligned}f_{q/N}^{(1)}(x, \mu) &= 2\text{Re}(\tilde{M}_a + \tilde{M}_b) + \tilde{M}_c - \frac{\alpha_s C_F}{4\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \delta(1-x) \\ &= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(\frac{2}{\epsilon_{\text{UV}}} + \frac{3}{2} + 2 \ln \frac{\mu}{\bar{n} \cdot p} \right) \delta(1-x) - 1 - x \right],\end{aligned}\tag{4.6}$$

where the last term in the first line comes from the self-energy of the fermion field ξ . The radiative correction $f_{q/N}^{(1)}$ contains the UV and IR divergences, furthermore there is the mixing of the UV and IR divergences. Due to the IR and mixed divergences, it is not appropriate to discuss the evolution of the collinear distribution function. Our factorization procedure removes the mixing of the UV and IR divergences as will be shown later.

In SCET_{II}, the zero-bin subtraction must be usoft, and the only difference from the soft zero-bin subtraction appears in \tilde{M}_b . The usoft zero-bin contribution for $M_{b,\text{us}}^{(0)}$ is given by

$$\begin{aligned}M_{b,\text{us}}^{(0)} &= 4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \delta(1-x) \int \frac{d^D l}{(2\pi)^D} \frac{\delta(l^2)}{n \cdot l (\bar{n} \cdot l + \delta_1)} \\ &= \frac{\alpha_s C_F}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(\frac{1}{\epsilon_{\text{UV}}} - \ln \frac{\delta_1}{\mu} \right),\end{aligned}\tag{4.7}$$

and the corresponding collinear contribution is given by

$$\tilde{M}_{b,\text{us}} = M_b - M_{b,\text{us}}^{(0)} = \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\delta(1-x) \left(-\frac{1}{\epsilon_{\text{UV}}} - \ln \frac{\mu}{\bar{n} \cdot p} \right) + \frac{x}{(1-x)_+} \right].\tag{4.8}$$

Note that $M_{a,\text{us}} = M_{a,s} = -M_{b,\text{us}}$, hence $M_{a,\text{us}} + M_{b,\text{us}} = 0$, meaning that there is no usoft zero-bin contributions in the PDF.

The PDF $\phi_{q/N}^{(1)}(x, \mu)$ at one loop is given by

$$\begin{aligned}\phi_{q/N}^{(1)}(x, \mu) &= 2\text{Re}(\tilde{M}_a + \tilde{M}_{b,\text{us}}) + \tilde{M}_c - \frac{\alpha_s C_F}{4\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \delta(1-x) \\ &= \frac{\alpha_s C_F}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(\frac{3}{2} \delta(1-x) + \frac{1+x^2}{(1-x)_+} \right).\end{aligned}\tag{4.9}$$

Unlike $f_{q/N}^{(1)}$, the radiative correction $\phi_{q/N}^{(1)}$ does not involve the mixed divergence. The result is the same as in full QCD. The IR divergence is absorbed in the nonperturbative part of $\phi_{q/N}$, and the UV divergent part yields the anomalous dimension of the PDF, which governs the renormalization group behavior.

The fact that the usoft zero-bin contributions in the PDF vanish is responsible for why full QCD results can be obtained from naive collinear contribution only. Of course, the full QCD results hold away from the endpoint region, and near the endpoint. It can be clearly explained in SCET. Away from the endpoint region, there is no intermediate scale and there is no need to construct effective theories successively. And there are no delta functions in the soft functions in Eqs. (3.1) and (3.7). Then the soft functions are just one, meaning that there are no soft contributions to all orders in α_s . And the radiative corrections of the collinear part at low energy scale are exactly those of the PDF given by Eq. (4.9). However, there is no way to avoid the steps described above near the endpoint region.

The relation between the collinear function $f_{q/N}$ and the PDF $\phi_{q/N}$ is given by

$$f_{q/N}(x, \mu) = \int_x^1 \frac{dz}{z} K_{qq}(z, \mu) \phi_{q/N}\left(\frac{x}{z}, \mu\right), \quad (4.10)$$

where $K_{qq}(z, \mu)$ is the difference between the zero-bin subtractions in the collinear matrix element between SCET_I and SCET_{II}. At one loop, it is given by

$$\begin{aligned} K_{qq}^{(1)}(z, \mu) &= 2\text{Re}(-M_{b,s}^{(0)} + M_{b,\text{us}}^{(0)}) \\ &= \frac{\alpha_s C_F}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\delta(1-z) \left(\frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu}{\bar{n} \cdot p} \right) - \frac{1}{(1-x)_+} \right]. \end{aligned} \quad (4.11)$$

We stress again, K_{qq} is not the matching coefficient between SCET_I and SCET_{II}. It contains IR divergence, and the mixed divergence too. There also appear IR and mixed divergences in the soft function. Combining these two, there will be neither IR nor mixed divergence, as is explicitly shown at one loop.

4.2 Collinear gluon distribution function and PDF

The collinear gluon distribution function is defined as

$$f_{g/N}(x, \mu) = \frac{1}{x \bar{n} \cdot P} \langle N(P) | \mathcal{B}_n^{\perp \mu a} \delta(x \bar{n} \cdot P - \bar{n} \cdot \mathcal{P}) \mathcal{B}_{n\mu}^{\perp a} | N(p) \rangle, \quad (4.12)$$

which is normalized to $f_{g/N}(x) = \delta(1-x)$ at tree level. The Feynman diagrams for the collinear gluon distribution function at one loop is shown in Fig. 3.

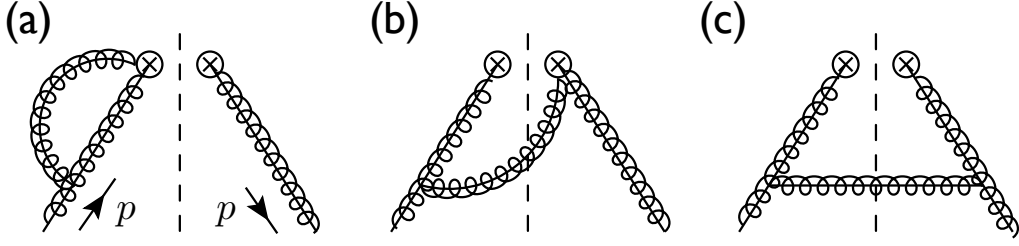


Figure 3. Feynman diagrams for collinear gluon distribution functions and PDF at one loop (a) virtual corrections, (b) and (c) real gluon emission. The mirror images of (a) and (b) are omitted.

The matrix elements in the background gauge are given as

$$\begin{aligned}
M_a &= 2ig^2 C_A \bar{n} \cdot p \delta(1-x) \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(l+p)^2} \left[\frac{1}{\bar{n} \cdot l - \delta_1} - \frac{1}{\bar{n} \cdot (l+p) + \delta_1} \right] \\
&= \frac{\alpha_s C_A}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \ln \frac{\delta_1}{\bar{n} \cdot p}, \\
M_b &= \frac{2\pi g^2 C_A}{x \bar{n} \cdot p} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \delta(l^2) \delta(\bar{n} \cdot l - (1-x)\bar{n} \cdot p) \\
&\quad \times \frac{[\bar{n} \cdot (l-p)]^2}{(l-p)^2} \left[\frac{\bar{n} \cdot (p-2l)}{\bar{n} \cdot p} - \frac{2\bar{n} \cdot p}{\bar{n} \cdot l + \delta_1} \right] \\
&= \frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(-\delta(1-x) \ln \frac{\delta_1}{\bar{n} \cdot p} + \frac{x}{(1-x)_+} + x(1-x) - \frac{x}{2} \right), \\
M_c &= \frac{8\pi g^2 C_A}{x \bar{n} \cdot p} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \delta(l^2) \delta(\bar{n} \cdot l - (1-x)\bar{n} \cdot p) \frac{1}{(n \cdot l)^2} \left(\frac{x^2}{2} n \cdot l \bar{n} \cdot p - \mathbf{l}_\perp^2 \right) \\
&= \frac{\alpha_s C_A}{\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(\frac{x}{2} + \frac{1-x}{x} \right). \tag{4.13}
\end{aligned}$$

The soft zero-bin contributions are given as

$$\begin{aligned}
M_a^{(0)} &= -\frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(\frac{1}{\epsilon_{UV}} - \ln \frac{\delta_1}{\mu} \right) \delta(1-x), \\
M_{b,s}^{(0)} &= \frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(-\delta(1-x) \ln \frac{\delta_1}{\bar{n} \cdot p} + \frac{1}{(1-x)_+} \right), \\
M_c^{(0)} &= 0. \tag{4.14}
\end{aligned}$$

The usoft zero-bin contribution only differs in the real gluon emission of M_b , and it is given as

$$M_{b,us}^{(0)} = \frac{\alpha_s C_A}{2\pi} \delta(1-x) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(\frac{1}{\epsilon_{UV}} - \ln \frac{\delta_1}{\mu} \right). \tag{4.15}$$

Therefore the matrix elements of the collinear gluon distribution function at one loop with

the soft zero-bin subtraction are given by

$$\begin{aligned}\tilde{M}_a &= M_a - M_a^{(0)} = \frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(\frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu}{\bar{n} \cdot p} \right) \delta(1-x), \\ \tilde{M}_{b,s} &= M_b - M_{b,s}(0) = \frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left(x(1-x) - \frac{x}{2} - 1 \right), \\ \tilde{M}_c &= M_c,\end{aligned}\tag{4.16}$$

and the usoft zero-bin subtraction $\tilde{M}_{b,\text{us}}$ is given by

$$\begin{aligned}\tilde{M}_{b,\text{us}} &= M_b - M_{b,\text{us}}^{(0)} \\ &= \frac{\alpha_s C_A}{2\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(-\frac{1}{\epsilon_{\text{UV}}} - \ln \frac{\mu}{\bar{n} \cdot p} \right) \delta(1-x) + \frac{x}{(1-x)_+} + x(1-x) - \frac{x}{2} \right].\end{aligned}\tag{4.17}$$

Combining all the ingredients, the collinear gluon distribution function at one loop is written as

$$\begin{aligned}f_{g/N}^{(1)}(x, \mu) &= 2\text{Re}(\tilde{M}_a + \tilde{M}_{b,s}) + M_c + \frac{\alpha_s \beta_0}{4\pi} \delta(1-x) \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \\ &= \frac{\alpha_s C_A}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(\frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu}{\bar{n} \cdot p} + \frac{11}{12} - \frac{n_f}{6N_c} \right) \delta(1-x) \right. \\ &\quad \left. + x(1-x) - 1 + \frac{1-x}{x} \right],\end{aligned}\tag{4.18}$$

where the last term in the first line is the self-energy correction of the gluon field in the background gauge, and β_0 is given by

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}n_f.\tag{4.19}$$

The gluon PDF $\phi_{g/N}$ is obtained by replacing $\tilde{M}_{b,s}$ with $\tilde{M}_{b,\text{us}}$ in Eq. (4.18), and is given by

$$\phi_{g/N}^{(1)}(x, \mu) = \frac{\alpha_s C_A}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(\frac{11}{12} - \frac{n_f}{6N_c} \right) \delta(1-x) + \frac{x}{(1-x)_+} + x(1-x) + \frac{1-x}{x} \right].\tag{4.20}$$

The one-loop correction to $\phi_{g/N}^{(1)}$ is the same as the result in full QCD for the same reason as in the case of the quark PDF. The initial-state jet function at one loop is given by

$$\begin{aligned}K_{gg}^{(1)}(x, \mu) &= 2\text{Re}(-M_{b,s}^0 + M_{b,\text{us}}^0) \\ &= \frac{\alpha_s C_A}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right) \left[\left(\frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu}{\bar{n} \cdot p} \right) \delta(1-x) - \frac{1}{(1-x)_+} \right].\end{aligned}\tag{4.21}$$

Compared to the initial-jet function with quarks in Eq. (4.11), $K_{gg}^{(1)}$ is the same except the color factor, satisfying $K_{qq}^{(1)}(x)/C_F = K_{gg}^{(1)}(x)/C_A$.

5 Final-state jet function

The final-state jet function for DIS in the \bar{n} direction is defined as

$$\sum_{X_{\bar{n}}} \chi_{\bar{n}} |X_{\bar{n}}\rangle \langle X_{\bar{n}}| \bar{\chi}_{\bar{n}} = \frac{\not{n}}{2} \int \frac{d^4 p_X}{(2\pi)^4} \bar{J}(p_X). \quad (5.1)$$

The jet function $\bar{J}(n \cdot p_X)$ is a function of $n \cdot p_X$ only. As defined in Eq. (2.32), the radiative corrections to $J(z, \mu) = Q\bar{J}(Q(1-z))/(2\pi)$ will be computed. It has been computed to two-loop order [27], but we will explicitly present the calculation at one loop to show how it can be computed like the collinear distribution function. However, the final states are not on the mass shell $p_X^2 \sim Q^2(1-z)$, the IR divergence is regulated by the offshellness p_X^2 . In the intermediate calculation, the IR poles appear, but in the final sum they cancel.

The Feynman diagrams for the final-state jet function at one loop are shown in Fig. 4. Fig. 4 (a) and its mirror image are the virtual corrections, and Fig. 4 (b) and (c) are the real gluon emissions. The virtual correction of the fermion self energy is omitted in the figure, but is added separately in the final calculation. Their matrix elements are given as

$$\begin{aligned} M_a &= -2ig^2 C_F \delta(1-z) \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{n \cdot (l+p)}{l^2 (l+p)^2 (n \cdot l + \delta_2)} \\ &= \frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(1 + \ln \frac{-\delta_2}{n \cdot p} \right), \\ M_b &= 8\pi^2 g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \frac{Q^2}{p^2} \int \frac{d^D l}{(2\pi)^D} \frac{n \cdot (p-l)}{(n \cdot l - \delta_2)} \delta(l^2) \delta((l-p)^2) \\ &= \frac{\alpha_s C_F}{2\pi} \left\{ \delta(1-z) \left[\left(\frac{1}{\epsilon_{IR}} + 2 \ln \frac{\mu}{n \cdot p} \right) \left(1 + \ln \frac{-\delta_2}{n \cdot p} \right) + 2 - \frac{\pi^2}{3} - \frac{1}{2} \ln^2 \frac{-\delta_2}{n \cdot p} \right. \right. \\ &\quad \left. \left. - \frac{1}{(1-z)_+} \left(1 + \ln \frac{-\delta_2}{n \cdot p} \right) \right] \right\}, \\ M_c &= 2\pi g^2 C_F (D-2) \frac{Q^3}{(p^2)^2} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{\mathbf{l}_\perp^2}{n \cdot (p-l)} \delta(l^2) \delta((l-p)^2) \\ &= \frac{\alpha_s C_F}{4\pi} \left[-\delta(1-z) \left(\frac{1}{\epsilon_{IR}} + 2 \ln \frac{\mu}{n \cdot p} + 1 \right) + \frac{1}{(1-z)_+} \right]. \end{aligned} \quad (5.2)$$

To be consistent with the idea of the zero-bin subtraction, the soft contributions should be subtracted from the naive collinear calculation. The zero-bin contributions in this case

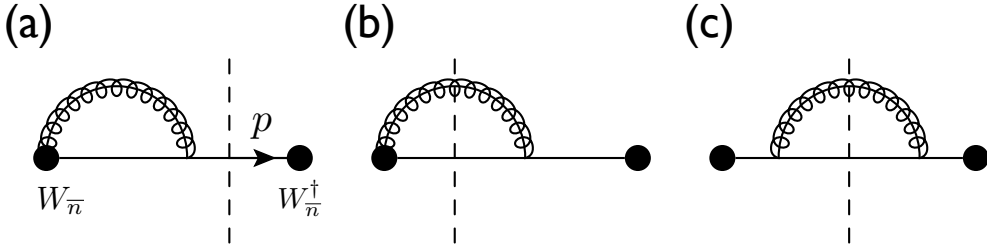


Figure 4. Feynman diagrams for the final-state jet functions in DIS at one loop (a) virtual corrections, (b) and (c) real gluon emission. The mirror images of (a) and (b) are omitted.

are easy to deduce, and the soft zero-bin subtraction is the appropriate procedure. The zero-bin contributions from Eq. (5.2) are given as

$$\begin{aligned}
M_a^{(0)} &= -\frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(1 + \ln \frac{-\delta_2}{n \cdot p} \right), \\
M_b^{(0)} &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon_{UV}\epsilon_{IR}} + \frac{1}{\epsilon_{IR}} \ln \frac{-\delta_2}{\mu} - \frac{1}{\epsilon_{UV}} \ln \frac{\mu}{n \cdot p} - \frac{1}{2} \ln^2 \frac{\mu^2}{-\delta_2 n \cdot p} - \frac{\pi^2}{12} \right) \right. \\
&\quad \left. + \frac{1}{(1-z)_+} \left(\frac{1}{\epsilon_{UV}} - \ln \frac{-\delta_2}{\mu} + \ln \frac{\mu}{n \cdot p} \right) - \left(\frac{\ln(1-z)}{(1-z)_+} \right) \right], \\
M_c^{(0)} &= 0,
\end{aligned} \tag{5.3}$$

where $M_c^{(0)}$ here is again subleading and is set to zero. The final result for the final-state jet function with the zero-bin subtraction is written as

$$\begin{aligned}
\tilde{M}_a &= M_a - M_a^{(0)} \\
&= \frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \left(\frac{1}{\epsilon_{UV}} + \ln \frac{\mu}{n \cdot p} + 1 \right), \\
\tilde{M}_b &= M_b - M_b^{(0)} \\
&= \frac{\alpha_s C_F}{2\pi} \left\{ \delta(1-z) \left[\frac{1}{\epsilon_{IR}} \left(\frac{1}{\epsilon_{UV}} + \ln \frac{\mu}{n \cdot p} + 1 \right) + \frac{1}{\epsilon_{UV}} \ln \frac{\mu}{n \cdot p} + 2 \ln \frac{\mu}{n \cdot p} \right. \right. \\
&\quad \left. \left. + \frac{1}{2} \ln^2 \frac{\mu^2}{(n \cdot p)^2} + 2 - \frac{\pi^2}{4} \right] - \frac{1}{(1-z)_+} \left(\frac{1}{\epsilon_{UV}} + 1 + 2 \ln \frac{\mu}{n \cdot p} \right) + \left(\frac{\ln(1-z)}{(1-z)_+} \right) \right\}.
\end{aligned} \tag{5.4}$$

Therefore the final-state jet function in DIS at one loop is written as

$$\begin{aligned}
J^{(1)}(z) &= 2\text{Re}(\tilde{M}_a + \tilde{M}_b) + M_c - \frac{\alpha_s C_F}{4\pi} \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \\
&= \frac{\alpha_s C_F}{2\pi} \left\{ \delta(1-z) \left[\frac{2}{\epsilon_{UV}^2} + \frac{1}{\epsilon_{UV}} \left(\frac{3}{2} + 2 \ln \frac{\mu^2}{Q^2} \right) + \frac{3}{2} \ln \frac{\mu^2}{Q^2} + \ln^2 \frac{\mu^2}{Q^2} + \frac{7}{2} - \frac{\pi^2}{2} \right] \right. \\
&\quad \left. - \frac{1}{(1-z)_+} \left(\frac{2}{\epsilon_{UV}} + 2 \ln \frac{\mu^2}{Q^2} + \frac{3}{2} \right) + 2 \left(\frac{\ln(1-z)}{(1-z)_+} \right) \right\},
\end{aligned} \tag{5.5}$$

where the last term in the first line comes from the virtual correction with the self-energy for the fermion field. In Eq. (5.5), $n \cdot p$ is replaced by Q , appropriate in DIS. This result is the same as the one in Ref. [15]. It is confirmed that the final-state jet function has only UV divergences after the zero-bin subtraction.

6 The soft kernels W and the renormalization group equation

Combining all the one-loop results, the factorization formulae in Eqs. (2.8) and (2.9) can be explicitly presented. The kernel W to one loop is obtained by plugging all the one-loop

results in Eqs. (2.21), (2.36) and (2.51). They are given as

$$\begin{aligned}
W_{\text{DY}}(z, \mu) &= \delta(1-z) \left[1 + \frac{\alpha_s C_F}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}^2} + \frac{2}{\epsilon_{\text{UV}}} \ln \frac{\mu}{Q} + 2 \ln^2 \frac{\mu}{Q} - \frac{\pi^2}{4} \right) \right] \\
&\quad + \frac{\alpha_s C_F}{\pi} \left[\frac{1}{(1-z)_+} \left(-\frac{2}{\epsilon_{\text{UV}}} - 4 \ln \frac{\mu}{Q} \right) + 4 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right], \\
W_{\text{Higgs}}(z, \mu) &= \delta(1-z) \left[1 + \frac{\alpha_s C_A}{\pi} \left(\frac{1}{\epsilon_{\text{UV}}^2} + \frac{2}{\epsilon_{\text{UV}}} \ln \frac{\mu}{Q} + 2 \ln^2 \frac{\mu}{Q} - \frac{\pi^2}{4} \right) \right] \\
&\quad + \frac{\alpha_s C_A}{\pi} \left[\frac{1}{(1-z)_+} \left(-\frac{2}{\epsilon_{\text{UV}}} - 4 \ln \frac{\mu}{Q} \right) + 4 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right], \\
W_{\text{DIS}}(z, \mu) &= \delta(1-z).
\end{aligned} \tag{6.1}$$

As claimed, the soft kernels are free of IR divergences. These kernels can be computed systematically using perturbation theory, and the anomalous dimensions can be obtained from Eq. (6.1) to describe the scaling behavior.

Remarkably the kernel for DIS becomes $\delta(1-z)$ to one loop since $S_{\text{DIS}}^{(1)} + K_{qq}^{(1)} = 0$. Therefore the inclusive DIS cross section near the endpoint consists of the hard part, the final-state jet function and the PDF $\phi_{q/N}$ at the scale $\mu < E$. It means that the soft contribution and the initial-state jet function cancel. Therefore the PDF satisfies the ordinary evolution equation, as in full QCD. If we do not combine the soft function and the initial-state jet function as was done in conventional factorization approach, the soft function and the collinear part $f_{q/N} = K_{qq} \otimes \phi_{q/N}$ include IR and mixed divergences. Then neither the soft function nor the collinear distribution function is physical, and it is meaningless to consider the evolution of the collinear distribution function. The solution to this problem is our approach. That is, we combine the soft function and the initial-state jet function to make an IR-finite kernel, which has no radiative corrections for DIS, and the remaining collinear part is the PDF $\phi_{q/N}$ gives the same result as in full QCD, given by Eq. (4.9). The difficulty in treating the evolution of the collinear part without the reorganization has been discussed in Refs. [14, 15, 28].

The fact that $W_{\text{DIS}} = \delta(1-z)$ is true to all orders in α_s , and it can be shown by a simple argument. The soft zero-bin contribution from $f_{q/N}$ is obtained by integrating out the momenta of order $Q\lambda$. This amounts to attaching a soft gluon to a collinear fermion, and making a loop with the soft gluon. This is exactly the procedure to obtain the eikonal form of the soft Wilson line and to calculate its loop correction. That is, the soft zero-bin contribution is the same as the soft function in DIS. On the other hand, the usoft zero-bin contributions vanish in $\phi_{q/N}$ to all orders in α_s . It is explicitly verified here at one loop, but if we look at Eqs. (3.1) and (3.7), the derivative term in the delta function is much smaller than $(1-z)$, hence can be neglected. Then the delta function can be pulled out, and the remaining usoft Wilson lines cancel. The usoft function becomes $\delta(1-z)$ to all orders in α_s . Therefore in the absence of the usoft contributions, the initial-state jet function, which is the negative value of the soft zero-bin contributions, always cancels the soft function to all orders in α_s in DIS.

In contrast, it is different in DY processes since the soft function involves an interaction between n and \bar{n} -collinear fermions, while the collinear part which interacts only within

each collinear sector does not produce the same soft interaction in the zero-bin limit. Note, however, that W_{DY} is still IR finite though the soft function involves the interaction between different collinear parts, and the initial-state jet function includes the interaction only in each collinear sector. The disparity between the soft function and the initial-state jet function becomes acute in multijet processes, and it will be interesting to see if the corresponding kernel will still remain IR finite in a more general case with multijets.

The factorized forms of the structure functions involving the kernels W have been already shown in Eqs. (2.20) and (2.35). Each factorized function has a nontrivial UV behavior, however when we combine all together, the structure functions and the scattering cross section should have no scale dependence.

$$\mu \frac{d}{d\mu} F_{\text{DY}}(\tau) = 0, \quad \mu \frac{d}{d\mu} F_1(x) = 0, \quad \mu \frac{d}{d\mu} \sigma_{\text{Higgs}} = 0. \quad (6.2)$$

Since all the elements in the factorization theorem are computed to one loop, the evolution of each quantity can be derived to next-to-leading logarithm accuracy.

The Wilson coefficients $C_{\text{DIS}}(Q^2, \mu)$ and $C_H(Q, \mu)$ are given by

$$\begin{aligned} C_{\text{DIS}}(Q^2, \mu) &= 1 + \frac{\alpha_s C_F}{4\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} - 3 \ln \frac{\mu^2}{Q^2} - 8 + \frac{\pi^2}{6} \right), \\ C_H(Q, \mu) &= 1 + \frac{\alpha_s C_A}{4\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} + \frac{7}{6} \pi^2 - 2i\pi \ln \frac{\mu^2}{Q^2} \right). \end{aligned} \quad (6.3)$$

Using the relation $C_{\text{DY}}(Q^2, \mu) = C_{\text{DIS}}(-Q^2, \mu)$, and from Eq. (2.41), the hard coefficients to one loop are given by

$$\begin{aligned} H_{\text{DIS}}(Q, \mu) &= 1 + \frac{\alpha_s C_F}{2\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} - 3 \ln \frac{\mu^2}{Q^2} - 8 + \frac{\pi^2}{6} \right), \\ H_{\text{DY}}(Q, \mu) &= 1 + \frac{\alpha_s C_F}{2\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} - 3 \ln \frac{\mu^2}{Q^2} - 8 + \frac{7\pi^2}{6} \right), \\ H_{\text{Higgs}}(Q, \mu) &= 1 + \frac{\alpha_s C_A}{2\pi} \left(-\ln^2 \frac{\mu^2}{Q^2} + \frac{7}{6} \pi^2 \right). \end{aligned} \quad (6.4)$$

From Eq. (6.4), the anomalous dimensions of the hard functions in DY, DIS and the Higgs production processes at one loop are given by

$$\begin{aligned} \gamma_{H_{\text{DY}}}(\mu) &= \mu \frac{d}{d\mu} H_{\text{DY}} = -\frac{\alpha_s C_F}{\pi} \left(2 \ln \frac{\mu^2}{Q^2} + 3 \right) = \gamma_{H_{\text{DIS}}}, \\ \gamma_{H_{\text{Higgs}}}(\mu) &= \mu \frac{d}{d\mu} H_{\text{Higgs}} = -\frac{2\alpha_s C_A}{\pi} \ln \frac{\mu^2}{Q^2}. \end{aligned} \quad (6.5)$$

In the Higgs production, there is also $|C_t(m_t, \mu)|^2$ in σ_0 , which is proportional to α_s^2 . To one loop, the anomalous dimension for $|C_t(m_t, \mu)|^2$ is given by

$$\gamma_{C_t} = \frac{d \ln |C_t|^2}{d \ln \mu} = -\frac{\alpha_s C_A}{\pi} \left(\frac{11}{3} - \frac{2n_f}{3N_c} \right), \quad (6.6)$$

which comes from the running of α_s at one loop

$$\mu \frac{\partial \alpha_s}{\partial \mu} = -\frac{\alpha_s^2}{2\pi} \beta_0 = -\frac{\alpha_s^2}{2\pi} C_A \left(\frac{11}{3} - \frac{2n_f}{3N_c} \right). \quad (6.7)$$

For the soft kernel W_{DY} , W_{Higgs} , and the PDF $\phi_{q/N}$, $\phi_{g/N}$, they satisfy the renormalization group equation

$$\begin{aligned}\mu \frac{d}{d\mu} W(x, \mu) &= \int_x^1 \frac{dz}{z} \gamma_W(z, \mu) W\left(\frac{x}{z}, \mu\right), \\ \mu \frac{d}{d\mu} \phi_{q/N}(x, \mu) &= \int_x^1 \frac{dz}{z} \gamma_q(z, \mu) \phi_{q/N}\left(\frac{x}{z}, \mu\right), \\ \mu \frac{d}{d\mu} \phi_{g/N}(x, \mu) &= \int_x^1 \frac{dz}{z} \gamma_g(z, \mu) \phi_{g/N}\left(\frac{x}{z}, \mu\right).\end{aligned}\tag{6.8}$$

The anomalous dimensions $\gamma_W(z, \mu)$ and $\gamma_{q,g}(z, \mu)$ are schematically given as

$$\mu \frac{d}{d\mu} Z(x, \mu) = - \int_x^1 \frac{dz}{z} \gamma(z, \mu) Z\left(\frac{x}{z}, \mu\right),\tag{6.9}$$

where Z is the counterterm and γ is the anomalous dimension for W or $\phi_{q/N}$, $\phi_{g/N}$. As discussed, W_{DIS} does not evolve.

From Eq. (6.1), the anomalous dimension of the kernel, W_{DY} , is given to one loop by

$$\gamma_{W_{\text{DY}}}(x, \mu) = \frac{\alpha_s C_F}{\pi} \left[2 \ln \frac{\mu^2}{Q^2} \delta(1-x) - \frac{4}{(1-x)_+} \right],\tag{6.10}$$

and the anomalous dimension $\gamma_{W_{\text{Higgs}}}$ for W_{Higgs} is obtained from γ_W by replacing C_F by C_A . From Eq. (4.9), the anomalous dimensions for the quark PDF $\phi_{q/N}$ and the gluon PDF are given by

$$\begin{aligned}\gamma_q(x, \mu) &= \frac{\alpha_s C_F}{\pi} \left[\frac{3}{2} \delta(1-x) + \frac{1+x^2}{(1-x)_+} \right] = \frac{\alpha_s}{\pi} P_{qq}(x) = \frac{\alpha_s}{\pi} P_{\bar{q}q}(x), \\ \gamma_g(x, \mu) &= \frac{\alpha_s}{\pi} 2N_c \left[\left(\frac{11}{12} - \frac{n_f}{6N_c} \right) \delta(1-x) + x(1-x) + \frac{x}{(1-x)_+} \right] = \frac{\alpha_s}{\pi} P_{gg}(x),\end{aligned}\tag{6.11}$$

where P_{qq} , $P_{\bar{q}q}$ and P_{gg} are the splitting kernels appearing in the DGLAP evolution equations for the PDF. The anomalous dimension for the final-state jet function is computed as

$$\gamma_J(x, \mu) = \frac{\alpha_s C_F}{\pi} \left[\left(2 \ln \frac{\mu^2}{Q^2} + \frac{3}{2} \right) \delta(1-x) - \frac{2}{(1-x)_+} \right] + \mathcal{O}(\alpha_s^2).\tag{6.12}$$

For each process, the sums of the anomalous dimensions near the endpoint $x \rightarrow 1$ are given as

$$\begin{aligned}\gamma_{H_{\text{DIS}}} \delta(1-x) + \gamma_q(x) + \gamma_J(x) &= 0, \\ \gamma_{H_{\text{DY}}} \delta(1-x) + \gamma_{W_{\text{DY}}}(x) + 2\gamma_q(x) &= 0, \\ (\gamma_{C_t} + \gamma_{H_{\text{Higgs}}}) \delta(1-x) + \gamma_{W_{\text{Higgs}}}(x) + 2\gamma_g(x) &= 0.\end{aligned}\tag{6.13}$$

Combining all these anomalous dimensions, we can see explicitly that Eq. (6.2) holds true to one loop near the endpoint.

7 Conclusion

The conventional factorization theorems have been successful in the sense that the effects of strong interactions at various stages have been satisfactorily separated to express scattering cross sections as convolutions of the high-energy part, the collinear and the soft parts. But there has remained a problem since the divergence structure is so intricate that the collinear and the soft parts still contain UV, IR, and mixed divergences. Now we have better understanding of the origins of the divergences, and this problem is taken care of by separating the soft modes consistently in the collinear part. As a result, the factorized parts no longer involve problematic IR or mixed divergences. With our new factorization scheme, factorization theorems have gained stronger grounds for the theoretical description of high-energy scattering.

Our factorization formula starts with a physical idea that the soft and collinear modes should be separated at higher loops, and employs the zero-bin subtraction to realize this idea. Our new factorization theorem emphasizes consistent separation of each mode at higher loops. In loop calculations, the soft contribution always encroaches on the collinear sector since the collinear loop momentum covers the soft region. To ensure the separation of the collinear and soft parts, the soft contribution in the collinear sector should be consistently removed from the collinear sector. Otherwise the overlap is bound to occur in every collinear loop calculation.

The consistent treatment of the zero-bin subtraction results in the appropriate divergence behavior. Without including the zero-bin contributions from the collinear part, the soft contribution contains not only the IR divergence but also the mixing of the IR and UV divergences. The mixed divergence is especially troublesome and it should be absent for the soft function to have physical meaning. The inclusion of the initial-state jet functions K_{qq} or K_{gg} removes this mixed divergence. In addition, it also changes the IR divergence to the UV divergences, and the kernels W are physically meaningful. If we naively put $\epsilon = \epsilon_{UV} = \epsilon_{IR}$ and identify the poles in ϵ as the UV poles, the soft function $S_{DY}^{(1)}$ in Eq. (3.4) is identical to the kernel W_{DY} in Eq. (6.1). It would be a good mnemonic to identify the UV divergence, but physically it does not make sense. Since we know the origin of the divergences, the UV and IR divergences can be systematically identified, and a physical quantity should be free of IR divergence.

Let us finally summarize the recipe for our factorization procedure. First, we write down the scattering cross section. Second, as in the conventional approach, it can be factorized into the hard, collinear and soft parts. In full QCD, they can be scattering amplitudes. In SCET, the hard part is obtained from the Wilson coefficient, the collinear and the soft parts are defined as the matrix elements of the relevant operators. Third, radiative corrections are computed. If there is an intermediate scale, the collinear distribution function with the soft zero-bin subtraction above the scale can be related to the PDF with the usoft zero-bin subtraction below the scale. The relation is expressed in terms of the initial-state jet function. Finally, we combine the soft function and the initial-state jet function to yield the soft kernel, which is IR finite.

Our factorization scheme gives a consistent field theoretic treatment of the UV and IR

divergences. This scheme has also been successfully applied to DY processes with small transverse momentum [18]. In this case the size of the transverse momentum offers the intermediate scale which distinguishes SCET_I and SCET_{II}. The initial-state jet function takes the form

$$f_{q/N}(x, \mathbf{k}_\perp, \mu) = \int_x^1 \frac{dz}{z} K_{qq}^T(\mathbf{k}_\perp, \mu) \phi_{q/N}\left(\frac{x}{z}, \mu\right), \quad (7.1)$$

where $f_{q/N}(x, \mathbf{k}_\perp, \mu)$ is the transverse-momentum-dependent collinear distribution function, and $K_{qq}^T(\mathbf{k}_\perp, \mu)$ is the initial-state transverse-momentum-dependent jet function. Combining the initial-state jet function with the transverse-momentum-dependent soft function also yields an IR-finite soft kernel. Therefore our factorization formalism can be applied to various high-energy processes. It remains to be seen whether this can be a general formalism for factorization proof. A research in this direction is in progress.

A Soft functions with offshellness

In the appendices, the soft functions and the collinear distribution function are computed in the regularization scheme with the offshellness of the external particles for the IR divergence. By computing the soft kernel and the PDF in this scheme, we also show the cancellation of the IR and mixed divergences.

For DIS, the soft function is defined as

$$S_{\text{DIS}}(z) = \frac{1}{N_c} \langle 0 | \text{tr} \left[Y_n^\dagger \tilde{Y}_{\bar{n}} \delta\left(1 - z + \frac{\bar{n} \cdot \mathcal{R}}{Q}\right) \tilde{Y}_{\bar{n}}^\dagger Y_n \right] | 0 \rangle. \quad (\text{A.1})$$

But here instead of Eq. (2.6) for the soft Wilson lines, the offshellness of the collinear particle from which the soft Wilson lines stem is inserted, and the soft Wilson lines are modified as

$$\begin{aligned} Y_n &= \sum_{\text{perm}} \exp \left[\frac{1}{n \cdot \mathcal{R} - \Delta_1 + i0} (-gn \cdot A_s) \right], \\ Y_n^\dagger &= \sum_{\text{perm}} \exp \left[-gn \cdot A_s \frac{1}{n \cdot \mathcal{R}^\dagger + \Delta_1 - i0} \right], \\ \tilde{Y}_{\bar{n}} &= \sum_{\text{perm}} \exp \left[\frac{1}{\bar{n} \cdot \mathcal{R} + \Delta_2 - i0} (-g\bar{n} \cdot A_s) \right], \\ \tilde{Y}_{\bar{n}}^\dagger &= \sum_{\text{perm}} \exp \left[-g\bar{n} \cdot A_s \frac{1}{\bar{n} \cdot \mathcal{R}^\dagger - \Delta_2 + i0} \right]. \end{aligned} \quad (\text{A.2})$$

The offshellness is given by $\Delta_1 = -p_1^2/\bar{n} \cdot p_1$, and $\Delta_2 = -p_2^2/n \cdot p_2$ where p_1 (p_2) is the n (\bar{n}) collinear momentum of the collinear particles with the corresponding soft Wilson lines to be attached. Note that the insertion of Δ_1 and Δ_2 looks similar to the rapidity regulator, but it is the regulator for the IR divergence. Though similar in form, their sources are distinct. If we put the offshellness explicitly, Δ_i and δ_i take different forms. In the soft Wilson line, Δ_i can be obtained from the offshellness of a single collinear particle where the soft gluons are attached. On the other hand, the rapidity regulator δ_i is obtained by the emission of the

n -collinear gluons from all the collinear or heavy particles in other directions. Therefore δ_i have complicated dependence on the offshellness of the other particles. Only in the back-to-back current, there exists a simple relation $\delta_1 = \Delta_2$, $\delta_2 = \Delta_1$.

In obtaining the dependence on the offshellness, we consider collinear particles or antiparticles from $-\infty$, or to ∞ , as considered in Ref. [20], and assign nonzero offshellness to the collinear particles. Here also arises the problem of gauge invariance. But it suffices to say that this is only an intermediate step to regulate IR divergences with the offshellness since the dependence of the offshellness is cancelled in the final results.

Here we employ the dimensional regularization for the UV divergence, and the IR divergence appears as logarithms of Δ_1 or Δ_2 . The virtual correction in Fig. 1 (a), and the real gluon emission in Fig. 1 (b) are given as

$$\begin{aligned}
M_{s,\text{DIS}}^a &= -2ig^2 C_F \delta(1-z) \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (n \cdot l - \Delta_1) (\bar{n} \cdot l - \Delta_2)} \\
&= -\frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{\Delta_1 \Delta_2} + \frac{1}{2} \ln^2 \frac{\mu^2}{\Delta_1 \Delta_2} + \frac{\pi^2}{4} \right), \\
M_{s,\text{DIS}}^b &= 4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{(n \cdot l + \Delta_1) (\bar{n} \cdot l + \Delta_2)} \delta(l^2) \delta\left(1-z - \frac{\bar{n} \cdot l}{Q}\right) \\
&= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon} \ln \frac{\Delta_2}{Q} - \ln \frac{\Delta_2}{Q} \ln \frac{\mu^2}{-p_1^2} + \frac{1}{2} \ln^2 \frac{\Delta_2}{Q} + \frac{\pi^2}{6} \right) \right. \\
&\quad \left. + \frac{1}{(1-z)_+} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p_1^2} \right) - \left(\frac{\ln(1-z)}{1-z} \right)_+ \right]. \tag{A.3}
\end{aligned}$$

In calculating these matrix elements, the following plus distribution functions are used.

$$\begin{aligned}
\frac{1}{1-z+\delta} &= -\delta(1-z) \ln \delta + \frac{1}{(1-z)_+}, \\
\frac{\ln(1-z)}{1-z+\delta} &= \delta(1-z) \text{Li}_2\left(-\frac{1}{\delta}\right) + \left(\frac{\ln(1-z)}{1-z} \right)_+, \tag{A.4}
\end{aligned}$$

where $\text{Li}_2(x)$ is the dilogarithmic function. The one-loop result for the soft function in DIS is given by

$$\begin{aligned}
S_{\text{DIS}}^{(1)}(z) &= 2\text{Re}(M_{s,\text{DIS}}^a + M_{s,\text{DIS}}^b) \\
&= \frac{\alpha_s C_F}{\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{-p_1^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{-p_1^2} - \frac{\pi^2}{12} \right) \right. \\
&\quad \left. + \frac{1}{(1-z)_+} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p_1^2} \right) - \left(\frac{\ln(1-z)}{1-z} \right)_+ \right]. \tag{A.5}
\end{aligned}$$

Note that the soft function does not depend on Δ_2 , as it should be. In DIS, the final state is described by the final-state jet function which depends on Δ_2 , but Δ_2 is not the IR cutoff. Instead it is related to the invariant jet mass of the final states. Therefore Δ_2 does not represent the IR divergence, and the soft function is independent of Δ_2 .

For DY process, the soft function is defined as

$$S_{\text{DY}}(z) = \frac{1}{N_c} \langle 0 | \text{tr} \left[Y_n^\dagger Y_{\bar{n}} \delta\left(1-z + \frac{2v \cdot \mathcal{R}}{Q}\right) Y_{\bar{n}}^\dagger Y_n \right] | 0 \rangle. \tag{A.6}$$

The soft Wilson lines $Y_{\bar{n}}$ and $Y_{\bar{n}}^\dagger$ are defined in the same way as Y_n and Y_n^\dagger except that n is replaced by \bar{n} , and Δ_1 by Δ_2 .

The virtual correction and the real gluon emission are given as

$$\begin{aligned}
M_{s,\text{DY}}^a &= -2ig^2 C_F \delta(1-z) \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2 (n \cdot l - \Delta_1) (\bar{n} \cdot l + \Delta_2)} \\
&= -\frac{\alpha_s C_F}{2\pi} \delta(1-z) \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{\Delta_1(-\Delta_2)} + \frac{1}{2} \ln^2 \frac{\mu^2}{\Delta_1(-\Delta_2)} + \frac{\pi^2}{4} \right), \\
M_{s,\text{DY}}^b &= 4\pi g^2 C_F \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^\epsilon \int \frac{d^D l}{(2\pi)^D} \frac{1}{(n \cdot l + \Delta_1) (\bar{n} \cdot l - \Delta_2)} \delta(l^2) \delta\left(1-z - \frac{2v \cdot l}{Q}\right) \\
&= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(\ln \frac{\Delta_1}{Q} \ln \frac{-\Delta_2}{Q} - \frac{\pi^2}{6} \right) - \frac{1}{(1-z)_+} \ln \frac{\Delta_1(-\Delta_2)}{Q^2} + 2 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right].
\end{aligned} \tag{A.7}$$

In the final stage of computing $M_{s,\text{DY}}^b$, there are two possibilities in taking the limit $\Delta_1, \Delta_2 \rightarrow 0$. That is, the limit $\Delta_1 \rightarrow 0$ can be approached first with Δ_2 fixed, and then the limit $\Delta_2 \rightarrow 0$ is taken. The limiting procedure can be reversed, however, the result is the same irrespective of the order of taking limits.

The soft function in DY process obtained by adding the hermitian conjugate of Eq. (A.7), and is given by

$$\begin{aligned}
S_{\text{DY}}^{(1)}(z) &= 2\text{Re}(M_{s,\text{DY}}^a + M_{s,\text{DY}}^b) \\
&= \frac{\alpha_s C_F}{\pi} \left\{ \delta(1-z) \left[-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \left(\ln \frac{\mu^2}{-p_1^2} + \ln \frac{\mu^2}{-p_2^2} - \ln \frac{\mu^2}{Q^2} \right) - \frac{5\pi^2}{12} \right. \right. \\
&\quad \left. \left. - \frac{1}{2} \ln^2 \frac{\mu^2}{-p_1^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{p_2^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} \right] \right. \\
&\quad \left. + \frac{1}{(1-z)_+} \left(\ln \frac{\mu^2}{-p_1^2} + \ln \frac{\mu^2}{-p_2^2} - 2 \ln \frac{\mu^2}{Q^2} \right) + 2 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right\}.
\end{aligned} \tag{A.8}$$

As can be seen again in Eqs. (A.5) and (A.7), the soft functions contain IR divergences as well as mixed divergences. Therefore the soft functions themselves are not physical.

B Collinear distribution functions with offshellness

The collinear distribution functions can also be evaluated with the offshellness for the IR regulator. The poles in ϵ are UV divergences. The naive collinear matrix elements are given as

$$\begin{aligned}
M_a &= \frac{\alpha_s C_F}{2\pi} \delta(1-z) \left[\frac{1}{\epsilon} \left(1 + \ln \frac{\delta}{Q} \right) + \ln \frac{\mu^2}{-p^2} - \frac{1}{2} \ln^2 \frac{\delta}{Q} + \ln \frac{\mu^2}{-p^2} \ln \frac{\delta}{Q} + 2 - \frac{\pi^2}{3} \right], \\
M_b &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon} \ln \frac{\delta}{Q} - \ln \frac{\delta}{Q} \ln \frac{\mu^2}{-p^2} + \frac{1}{2} \ln^2 \frac{\delta}{Q} + \frac{\pi^2}{6} \right) \right. \\
&\quad \left. + \frac{z}{(1-z)_+} \left(\frac{1}{\epsilon} - \ln z + \ln \frac{\mu^2}{-p^2} \right) - z \left(\frac{\ln(1-z)}{(1-z)} \right)_+ \right], \\
M_c &= \frac{\alpha_s C_F}{2\pi} (1-z) \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} - 2 - \ln z(1-z) \right).
\end{aligned} \tag{B.1}$$

Here we put $\bar{n} \cdot p = Q$, and δ is the rapidity regulator in the collinear Wilson line. If δ is not used, IR poles in ϵ_{IR} appear instead, and they also cancel.

The soft zero-bin contributions are given as

$$\begin{aligned} M_a^{(0)} &= -\frac{\alpha_s C_F}{2\pi} \delta(1-z) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2 Q}{-p^2 \delta} + \frac{1}{2} \ln^2 \frac{\mu^2 Q}{-p^2 \delta} + \frac{\pi^2}{4} \right], \\ M_{b,s}^{(0)} &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon} \ln \frac{\delta}{Q} - \ln \frac{\delta}{Q} \ln \frac{\mu^2}{-p^2} + \frac{1}{2} \ln^2 \frac{\delta}{Q} + \frac{\pi^2}{6} \right) \right. \\ &\quad \left. + \frac{1}{(1-z)_+} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} \right) - \left(\frac{\ln(1-z)}{1-z} \right)_+ \right], \\ M_c^{(0)} &= 0. \end{aligned} \tag{B.2}$$

Therefore the collinear contributions with the soft zero-bin subtractions are given as

$$\begin{aligned} \tilde{M}_a &= M_a - M_a^{(0)} \\ &= \frac{\alpha_s C_F}{2\pi} \delta(1-z) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(1 + \ln \frac{\mu^2}{-p^2} \right) + \ln \frac{\mu^2}{-p^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{-p^2} + 2 - \frac{\pi^2}{12} \right], \\ \tilde{M}_{b,s} &= M_b - M_{b,s}^{(0)} \\ &= \frac{\alpha_s C_F}{2\pi} \left[-\left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} \right) - \frac{z \ln z}{(1-z)_+} + \ln(1-z) \right], \\ \tilde{M}_c &= M_c = \frac{\alpha_s C_F}{2\pi} (1-z) \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} - 2 - \ln z(1-z) \right). \end{aligned} \tag{B.3}$$

The collinear quark distribution function at one loop is given as

$$\begin{aligned} f_{q/N}^{(1)}(z, \mu) &= 2\text{Re}(\tilde{M}_a + \tilde{M}_b) + M_c + \delta(1-z)(Z_\xi^{(1)} + R_\xi^{(1)}) \\ &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(\frac{2}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{2}{\epsilon} \ln \frac{\mu^2}{-p^2} + \frac{3}{2} \ln \frac{\mu^2}{-p^2} + \ln^2 \frac{\mu^2}{-p^2} + \frac{7}{2} - \frac{\pi^2}{6} \right) \right. \\ &\quad \left. - \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} \right) (1+z) - \frac{1+z^2}{(1-z)_+} \ln z - 2(1-z) + (1+z) \ln(1-z) \right], \end{aligned} \tag{B.4}$$

where $Z_\xi^{(1)}$ is the counterterm and $R_\xi^{(1)}$ is the residue in the self-energy of the fermion ξ_n at one loop and they are given as

$$Z_\xi^{(1)} = -\frac{\alpha_s C_F}{4\pi} \frac{1}{\epsilon}, \quad R_\xi^{(1)} = -\frac{\alpha_s C_F}{4\pi} \left(1 + \ln \frac{\mu^2}{-p^2} \right). \tag{B.5}$$

The usoft zero-bin contribution differs only in the real gluon emission, which is given as

$$\begin{aligned} M_{b,\text{us}}^{(0)} &= \frac{\alpha_s C_F}{2\pi} \delta(1-z) \left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(\ln \frac{\mu^2}{-p^2} - \ln \frac{\delta}{Q} \right) + \frac{\pi^2}{4} \right. \\ &\quad \left. + \frac{1}{2} \ln^2 \frac{\mu^2}{-p^2} - \ln \frac{\mu^2}{-p^2} \ln \frac{\delta}{Q} - \frac{1}{2} \ln^2 \frac{\delta}{Q} \right]. \end{aligned} \tag{B.6}$$

The corresponding collinear part with the usoft zero-bin subtraction is given as

$$\begin{aligned}\tilde{M}_{b,\text{us}} &= M_b - M_{b,\text{us}}^{(0)} \\ &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(-\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{-p^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{-p^2} - \frac{\pi^2}{12} \right) \right. \\ &\quad \left. + \frac{z}{(1-z)_+} \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} - \ln z \right) - z \left(\frac{\ln(1-z)}{1-z} \right)_+ \right].\end{aligned}\tag{B.7}$$

And the one-loop correction to the PDF is given as

$$\begin{aligned}\phi_{q/N}^{(1)}(z, \mu) &= \frac{\alpha_s C_F}{2\pi} \left[\delta(1-z) \left(\frac{3}{2\epsilon} + \frac{3}{2} \ln \frac{\mu^2}{-p^2} + \frac{7}{2} - \frac{\pi^2}{3} \right) + \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} \right) \frac{1+z^2}{(1-z)_+} \right. \\ &\quad \left. - 2(1-z) - (1+z^2) \frac{\ln z}{(1-z)_+} - (1+z^2) \left(\frac{\ln(1-z)}{1-z} \right)_+ \right].\end{aligned}\tag{B.8}$$

The initial-state jet function can be written as

$$\begin{aligned}K_{qq}(z, \mu) &= 2\text{Re}(-M_{b,s}^{(0)} + M_{b,\text{us}}^{(0)}) \\ &= \frac{\alpha_s C_F}{\pi} \left[\delta(1-z) \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{-p^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{-p^2} + \frac{\pi^2}{12} \right) \right. \\ &\quad \left. - \left(\frac{1}{\epsilon} + \ln \frac{\mu^2}{-p^2} \right) \frac{1}{(1-z)_+} + \left(\frac{\ln(1-z)}{1-z} \right)_+ \right].\end{aligned}\tag{B.9}$$

To one loop, the kernel W are given as

$$\begin{aligned}W_{\text{DIS}}^{(1)} &= 0, \\ W_{\text{DY}}^{(1)} &= \frac{\alpha_s C_F}{\pi} \left[\delta(1-z) \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} + \frac{\pi^2}{4} \right) \right. \\ &\quad \left. - \left(\frac{2}{\epsilon} + 2 \ln \frac{\mu^2}{Q^2} \right) \frac{1}{(1-z)_+} + 4 \left(\frac{\ln(1-z)}{1-z} \right)_+ \right].\end{aligned}\tag{B.10}$$

In this calculational scheme, the kernels W are also IR finite. As in the case with the dimensional regularization, $W_{\text{DIS}}^{(1)} = 0$. However, $W_{\text{DY}}^{(1)}$ is the same as the result with the dimensional regularization except the π^2 term. The source of the disparity can be seen from Eq. (A.8). There is a term $\ln^2(\mu^2/p_2^2)$ in S_{DY} , while there is $\ln^2(\mu^2/-p_2^2)$ in K_{qq} . Therefore the IR divergence cancels as expected, but there is a remnant of π^2 . It results in the difference of the term with π^2 . If we take the limit $p_1^2, p_2^2 \rightarrow 0$, it corresponds to the dimension regularization limit for the IR divergence. Then the logarithms turn into poles in ϵ_{IR} , and there is no additional π^2 term involved. However, if we strictly keep the signs of the offshellness, this additional factor of π^2 appears. If there were only single logarithms, this ambiguity does not occur. We have not been able to confirm whether the difference is due to the scheme dependence and further consideration is needed.

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